Delineation of Integer Solutions to Homogeneous Bi-quadratic Equation with Four Unknowns

$$x^4 + y^4 + z^4 = 18w^4$$

Dr. N. Thiruniraiselvi[1*], Dr. M.A.Gopalan[2]

[1] Assistant Professor, Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Samayapuram, Trichy- 621 112, Tamil Nadu, India.

[2] Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002,Tamil Nadu, India.

Abstract

The main thrust of this paper is to search for various choices of integer solutions to the homogeneous bi-quadratic equation with four unknowns given by $x^4 + y^4 + z^4 = 18w^4$.

Keywords: Homogeneous bi-quadratic, Bi-quadratic with four unknowns, Integer solutions.

I.INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context, one may refer [6-24] for various problems on the biquadratic Diophantine equations with four variables. However, often we come across homogeneous biquadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the homogeneous equation with four unknowns given by $x^4 + y^4 + z^4 = 18w^4$ is analyzed for obtaining different sets of non-zero distinct integer solutions through employing the linear transformations.

II. METHOD OF ANALYSIS

The homogeneous bi-quadratic equation with four unknowns to be solved is

$$x^4 + y^4 + z^4 = 18w^4 \tag{1}$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = 2v, u \neq v \neq 0 \tag{2}$$

in (1) leads to

$$u^2 + 3v^2 = 3w^2 \tag{3}$$

Solving (3) through different approaches, the values of u, v, w are obtained.

In view of (2), the corresponding values of x, y, z satisfying (1) are found.

Approach 1

Assume

$$w = a^2 + 3b^2 \tag{4}$$

Express the integer 3 on the R.H.S. of (3) as the product of complex conjugates

as below

$$3 = \frac{(3 + i\sqrt{3})\ (3 - i\sqrt{3})}{4} \tag{5}$$

Substituting (4) & (5) in (3) and employing the factorization method, consider

$$u + i\sqrt{3}\,v = \frac{(3 + i\sqrt{3})}{2}\ (a + i\sqrt{3}\,b)^2 \tag{6}$$

Equating the real and imaginary parts in (6) ,the values of u, v are obtained.

In view of (2), one has

$$x = 2(a^2 - 3b^2),\ y = (a^2 - 3b^2 - 6ab),\ z = (a^2 - 3b^2 + 6ab) \tag{7}$$

Thus, (4) and (7) represent the integer solutions to (1).

Note 1

It is observed that, in addition to (5), the integer 3 is also expressed as

$$3 = \frac{(12 + i\sqrt{3})\ (12 - i\sqrt{3})}{49}$$

Following the above analysis, a different set of integer solutions to (1) is obtained.

Approach 2

Rewrite (3) as

$$u^2 = 3(w^2 - v^2) \tag{8}$$

The R.H.S. of (8) is a perfect square when

$$w = 2s^2 + 2s + 2 \tag{9}$$

$$v = 2s^2 + 2s - 1 \tag{10}$$

In view of (8) , one has

$$u = 6s + 3 \tag{11}$$

Substituting (10) & (11) in (2) ,it is seen that

$$x = 2s^2 + 8s + 2 \,, y = -2s^2 + 4s + 4, z = 4s^2 + 4s - 2 \tag{12}$$

Thus, (9) & (12) give the integer solutions to (1).

Note 2

It is worth to see that the R.H.S. of (8) is also a perfect square when

$$w = 6s^2 + 6s + 2$$

$$v = 6s^2 + 6s + 1$$

In this case, the corresponding values of x, y, z are given by

$$x = 6s^2 + 12s + 4 \,, y = -6s^2 + 2, z = 12s^2 + 12s + 2$$

Approach 3

Consider (3) as

$$3w^2 - u^2 = 3v^2 \tag{13}$$

Assume

$$v = 3a^2 - b^2 \tag{14}$$

Write the integer 3 on the R.H.S. of (13) as the product of irrational conjugates as follows

$$3 = (2\sqrt{3} + 3)(2\sqrt{3} - 3) \tag{15}$$

Substituting (14) & (15) in (13) and applying the factorization method,

Consider

$$\sqrt{3}\,w + u = (2\sqrt{3} + 3)\,(\sqrt{3}\,a + b)^2 \tag{16}$$

Equating the rational and irrational parts in (16) , we have

$$u = 3(3a^2 + b^2) + 12ab \tag{17}$$

$$w = 2(3a^2 + b^2) + 6ab \tag{18}$$

Using (14) & (17) in (2) , we have

$$x = 12a^2 + 2b^2 + 12ab,\ y = 6a^2 + 4b^2 + 12ab,\ z = 6a^2 - 2b^2 \tag{19}$$

Thus, (18) & (19) represent the integer solutions to (1).

Note 3

It is observed that, in addition to (15) , the integer 3 is also expressed as

$$3 = (7\sqrt{3} + 12)(7\sqrt{3} - 12)$$

Following the above analysis, a different set of integer solutions to (1) is obtained.

III. CONCLUSION

In this paper, an attempt has been made to determine non-zero distinct integer solutions to the homogeneous biquadratic equation with four unknowns given by $x^4 + y^4 + z^4 = 18w^4$. The researchers in this field may search for other sets of integer solutions to the equation under consideration and other forms of biquadratic equations with four or more variables.

REFERENCES:

[1] L.E. Dickson, History of Theory of Numbers, Vol.11, Chelsea publishing company, New York (1952).

[2] L.J. Mordell, Diophantine equations, Academic press, London (1969).

[3] Carmichael, R.D., The theory of numbers and Diophantine Analysis, Dover publications, New York (1959).

[4] Telang, S.G., Number theory, Tata Mc Graw Hill publishing company, New Delhi (1996).

[5] Nigel, D. Smart, The Algorithmic Resolutions of Diophantine Equations, Cambridge University press, London (1999).

[6] M.A. Gopalan, V. Pandichelvi On the Solutions of the Biquadratic equation $(x^2 - y^2)^2 = (z^2 - 1)^2 + w^4$ paper presented in the international conference on Mathematical Methods and Computation, Jamal Mohammed College, Tiruchirappalli, July 24-25, 2009.

[7] K. Meena, S. Vidhyalakshmi, M.A. Gopalan, S. Aarthy Thangam, On the bi-quadratic equation with four unknowns $x^3 + y^3 = 39zw^3$, International Journal of Engineering Research Online, Vol. 2(1), 57-60, 2014.

[8] On the homogeneous bi-quadratic equation with four unknowns $x^4 + y^4 + z^4 = 32w^4$, Scholars Bulletin, Vol.1(7), 177-182, Oct 2015

[9] On the homogeneous bi-quadratic equation with five unknowns $x^4 - y^4 = 40(z+w)p^3$, International Journal of Current Research, Vol.7(07), Pp.18435-18438, July 2015

[10] A.Vijayasankar, Sharadha Kumar, M.A.Gopalan, "On the Non-Homogeneous Bi-Quadratic Equation with Four Unknowns $8xy + 5z^2 = 5w^4$", Jouranl of Xi'an University of architecture & Technology, 12(2), 2020, Pp:1108-1115.

[11] S. Vidhyalakshmi, T. Mahalakshmi, M.A. Gopalan, A Search for Integral solutions to the Ternary Bi-Quadratic Equation $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = (x+y)^2 + 1 + z^2$, *Turkish Journal of Computer and Mathematics Education*, **12** (7), (2021), 484-495.

[12] S.Vidhyalakshmi, M.A.Gopalan, On Finding integer solutions to Non-Homogeneous Ternary Bi-Quadratic Equation $3(x^2 + y^2) - 2xy = 11z^4$, *International journal of Novel Research in Physics, Chemistry and Mathematics* , **9** (2), (2022), 23-28.

[13] S. Vidhyalakshmi, M.A. Gopalan, On the Non-Homogeneous Ternary Bi-Quadratic $xz(x+z)=2y^4$, *International Research Publication and* Reviews, **3**, (2022), 3465-3469.

[14] S.Vidhyalakshmi, M.A. Gopalan, On the Non-Homogeneous Ternary Bi-Quadratic equation $8xz(x+z)=15y^4$, *International Research Journal of Moderization in Engineering Technology and Science (IRJMETS)*, **04** (07), (2022), 3623-3625.

[15] S. Vidhyalakshmi, M.A. Gopalan, On the Non-Homogeneous Ternary Bi-Quadratic equation $xz(x-z)=y^4$, *International Research Journal of Education and Technology*, **04** (07), (2022), 232-237.

[16] S. Vidhyalakshmi, M.A. Gopalan, On Non-Homogeneous Ternary Bi-Quadratic Equation $11(x+y)^2=4(xy+11z^4)$, *Journal of Multidisciplinary Engineering Science and Research (JMESR)*,**1** (1), (2022), 8-10.

[17] S. Vidhyalakshmi, M.A. Gopalan, On finding integer solutions to Non-Homogeneous Ternary Bi-Quadratic equation $x^2+3y^2=31z^4$, *International Journal of Multidisciplinary Research and Growth Evaluation*, **03** (04), (2022), 319-327.

[18] S. Vidhyalakshmi, M.A. Gopalan, On Non-Homogeneous Ternary Bi-Quadratic Equation $4xz(x+z)=5y^4$, *International Journal of Research Publication and Reviews*, **03** (08), (2022), 443-447.

[19] S. Vidhyalakshmi, M.A. Gopalan, On Non-Homogeneous Ternary Bi-Quadratic Equation $5(x^2-y^2)+2(x+y)=12z^4$, *International Research Journal of Moderlization in Engineering Technology and Science*, **04**(08), (2022), 425-429.

[20] S. Vidhyalakshmi, M.A. Gopalan, On Non-Homogeneous Ternary Bi-Quadratic Equation $2xz(x-z)=y^4$, *International Journal of Research publication and Rewiews,* **08** (08), (2022), 187-192.

[21] S. Vidhyalakshmi, M.A. Gopalan, On finding integer solution to Non-Homogeneous Ternary Bi-Quadratic equation $5(x^2+y^2)-2xy=140z^4$, *International Journal of Engineering Inventions*, **11** (08), (2022), 01-04.

[22] S. Vidhyalakshmi, J. Shanthi, M.A. Gopalan, Observation on the Non-Homogeneous Biquadratic Equation with five unknowns $(x^4-y^4)=10(z+w)p^2$, *Vidyabharati International Interdisciplinary Research Journal*, Special Issue on Recent Research Trends in Management, Science and Technology, (2021), 1048-1053.

[23] S.Vidhyalakshmi, M.A.Gopalan, On Homogeneous Bi-Quadratic Diophantine Equations with Five Unknowns $x^4-y^4=5^{2n}(z^2-w^2)T^2$, *International Journal of Engineering Inventions*, **11**(3), (2022), 293-298.

[24] S. Vidhyalakshmi, M.A. Gopalan, On finding general form of Integral solution to the Quinary Homogeneous Bi Quadratic equation $(x+y)(x^3+y^3)=\sim(z^2-w^2)\circ^2$, *International journal of Research publication and Reviews*, **03** (09), (2022), 1360-1363.