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An Innovative Fractional Integral Transform for the Solution of Well Known Partial Differential Equations

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Abstract

Our research introduced a novel and versatile fractional integral transform capable of solving a variety of integral and differential equations of both integer and fractional order. Furthermore, convergence of proposed transform and, findings of some other important application on different functions. We will derive the partial derivatives of different orders with variable t and x. Our main work encompasses particular solution of some well-known linear first and second order partial differential equations i.e., Laplace, Wave, One-dimensional homogeneous Heat and Telegraphers equations.

keyword Integral transform; Fractional order transform; Partial differential equations; Fractional differential equations; Integral equations. MSC codes: 31B10, 44A10, 26A33.

1 Introduction

This section reviews the previous research and provides the background information that sets the context for the current study. Mathematics is as old as the humanity on earth are, and it remains a novel field with a lot of potential for research. In history, individuals first solve the algebraic equations before progressing to differential equations. A a differential equation (DE) can be used as a mathematically model of phenomena, experiments, observations, or theories [39]. DE serves as powerful tools for modeling real-world problems in a scientific manner. Even a single DE can be used as a mathematical model for diverse physical systems, such as electrical series circuits and vibrating spring mass systems, which are almost identical. However, integral transforms come into play for certain complex problems that DE cannot easily address. A mathematical model of a physical system with a continuous or discontinuous function is easy to understand; nevertheless, when we consider a system with a piecewise continuous and periodic function,

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then transformation activates to solve the problem accurately. These transforms convert the problems into easily solvable algebraic equation forms, resulting in precise solutions [38, 39]. The history of integral transforms traces back to 1790, when a French mathematician and astronomer, Pierre-Simon Marquis de Laplace, introduced the Laplace transform while working on the probability theorem. And its versatility was shown in the work of Heaviside, an ordinary differential equation (ODE) of an electrical engineering problem. In 1801, Fourier introduced the Fourier transform, a special case of Laplace's introduction of the Laplace transform to explain heat flow. It can be used in chemistry, astronomy, medicine, engineering, and physics. In mathematics, it deals with the calculation of electrical signals to mechanical vibrations and from wave propagation to diffusion and Laplace equations.[38] Since then, numerous mathematicians have proposed a series of integral transforms, resulting in the development of many general and bilateral integral transforms to date. Recently, in 2022, Kuffi and Maktoof [1] presented a new integral transform derived from the Fourier transform, specifically designed for solving ordinary and partial differential equations in the t-domain. Additionally, Patil et al. [2] introduced the HY transform, tailored for addressing problems related to exponential growth. Kumar et al. [3] presented the Rishi transform for effectively handling linear Volterra integral equations of the first kind (LVIEFK). Furthermore, Upadhyaya et al. [4] presented an updated form of the Upadhyaya transform, defining it in one dimension and elucidating its generalization in n-dimensions. Over the past decade, various researchers [5, 16, 17, 18, 24] have proposed new integral transforms to address differential and integral equations of integer order with constant and variable coefficients [25, 26, 27].

More than one conformable transform also came into view to solve fractional differential equations and fractional calculus-related problems. Fractional order differential equations are effectively addressed using fractional integral transforms, which represent updated versions of traditional integral transforms. In 2019, Manjarekar and Bhadane [28] introduced the generalized fractional Elzaki-Tarig transform as a powerful tool for solving fractional order differential equations. They successfully obtained analytical solutions for the given fractional order differential equations. In a similar vein, Jafari [16] proposed a novel general Laplace-type integral transform called the "Jafari transform." This transform proves to be highly effective in solving fractional order integral transforms, Volterra integral transforms, and higher order ordinary differential equations (ODEs). By providing a versatile approach, the Jafari transform contributes significantly to the solution of complex problems involving fractional calculus. A lot of real-world problems ranging from engineering to medical science and any others are solved by using these wonderful integral transforms combined with other amazing techniques [29, 33].

Many practical problems in science and engineering give rise to partial differential equations (PDE) when designed mathematically. Heat, wave, Laplace, telegraph, and Burger's equations are a few well-known PDEs. In ODEs the solution can be found easily with ICs and without finding a general solution, and in PDEs transformations reduces the multi-variable problems to single-variable ones [38]. Mathematical models of population growth, noise in communication systems, traffic flow and telephonic traffic, radioactive decay, chemical reactions, current in series circuits, financial mathematics, and many other applications use first-order differential equations. A two-variable linear second-order partial differential equation (PDE) can either be classified as elliptic, parabolic, or hyperbolic. Laplace equation is an example of an elliptical PDE used to explain the steady state of a physical system [38, 39].

The heat equation is a form of parabolic PDE that describes a diffusional state, whereas a wave equation, a hyperbolic PDE, depicts a physical system having a vibrational state. The Telegraphs equation is a hyperbolic partial differential equation that predicts structural vibrations. The Laplace equation governs a wide range of physical phenomena, including steady heat conduction, the irrotational flow of an ideal fluid, the distribution of magnetic field dispersion, and so on. The Heat equation used in steady-state heat conduction, diffusion process, geophysics, financial mathematics, material sciences, and many other fields. Wave equations are seen in different scenarios, such as propagation of sound waves, surface waves in earthquakes, sea waves, etc. The telegrapher's equation is used in various physical and biological processes like pulsed blood flow in the arteries, random walks of humans and insects, electrical signals in transmission lines, and so on [36, 37, 38, 39].

1.1 Motivation

Differential equations are difficult to compute as compared to algebraic equations. Integral transforms are used to convert differential equations into algebraic equations which makes the solution process of differential equations more easier and more effective. A new integral transform to solve the differential equation for solving the differential equations of both integer and fractional order has been developed. The important features of new transform are

Proposed transform is the most generalized Laplace type integral transform.

Till now no transform is given which can switch between both integer and fraction order. This transform can be of integer order or fractional order for different values of α.

The proposed transform can also use to handle the differential and integral equations of any order type and order either they are classical or fractional.

Additionally, we looked into the particular solution of some highly recognized partial differential equations using the proposed transform.

This paper consists of five sections. Section 2 developed for the new fractional integral transform. In Section 3, we will apply transform on partial derivatives and will showcase different properties of integral transform. Section 4 is for application of the transform to the partial differential equations along with graphical view. Conclusions are drawn in Section 5.

2 Definition and existence of the new transform

Here we introduce the most general integral transform, which includes fractional order integral transforms and almost all of the integral transforms that are currently known within the Laplace transform family.

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Let $f(t)$ be an integrable function defined for all $t \geq 0$, $\gamma(t)$ is a normalization function, $\phi^{\alpha}(s) = 0$ and $\psi(s)$ are positive real functions, then the new integral transform is defined as

$$S\{f(t);s\} = F(x,s) = \phi^{\alpha}(s)\int_0^{\infty} \gamma(t)e^{-\psi(s)t}\ f(x,t)dt. \tag{2.1}$$

Provided that the integral exists for some $\psi(s)$ and inverse of the transform is the original function $f(t) = S^{-1}\{F(s)\}$.

Theorem 2.1. *Convergence of the transform: If $f(x,t)$ is a piecewise continuous function and is of exponential order θ and satisfies $|\gamma(t)f(x,t)| \leq Me^{-\theta t}$ then $S\{f(x,t);s\}$ exists $\forall\ \psi(s) > \theta$.*

Proof. Let $f(x,t)$ be a piecewise continuous function that is of exponential order θ and satisfies $|\gamma(t)f(t)| \leq Me^{-\theta t}$ then $S\{f(x,t);s\}$ exists $\forall\ \psi(s) > \theta$,

$$\|S\{f(x,t);s\}\| = \left|\phi^{\alpha}(s)\int_0^{\infty}\gamma(t)e^{-\psi(s)t}f(x,t)dt\right| \leq \phi^{\alpha}(s)\int_0^{\infty}e^{-\psi(s)t}\left|\gamma(t)f(x,t)\right|dt,$$

$$\leq \phi^{\alpha}(s)\int_0^{\infty}Me^{\theta t}e^{-\psi(s)t}dt \leq \frac{-M\phi^{\alpha}(s)}{(\theta-\psi(s))} = \frac{M\phi^{\alpha}(s)}{\psi(s)-\theta}.$$

Which completes the proof. □

3 Transform on partial derivatives

The objective of this section is to apply proposed transform on partial derivatives, that provide advantages in solving differential equations. Furthermore, we have provided a summary of the results obtained by applying the new integral transform to various functions in Table 1 under the conditions $\gamma(t) = 1$ and $t \geq 0$.

3.1 Transform on Partial derivatives with variable t

In this subsection, we apply the transform on Partial derivatives w.r.t variable t.

Theorem 3.1. *First Partial Derivative: If $S\{f(x,t);s\} = F(x,s)$, then*

$$\left\{\frac{\partial f}{\partial t}(x,t)\right\} = \psi(s)F(x,s) - \phi^{\alpha}(s)f(x,0).$$

Proof. Consider the function $S\{f(x,t);s\} = F(x,s)$, and after applying transform (2.1), considering $\gamma(t) = 1$, we have

$$S\left\{\frac{\partial f}{\partial t}(x,t)\right\} = \phi^{\alpha}(s)\int_0^{\infty}e^{-\psi(s)t}\frac{\partial f}{\partial t}(x,t)dt.$$

Using integration by parts, we have

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$$S\left\{\frac{\partial f}{\partial t}(x,t)\right\}=\phi^{\alpha}(s)[f(x,t)e^{-\psi(s)t}]_0^{\infty}+\phi^{\alpha}(s)\psi(s)\int_0^{\infty}f(x,t)e^{-\psi(s)t}dt.$$
$$=\phi^{\alpha}(s)(0-f(x,0))+\psi(s)F(x,s).$$

This implies

$$S\left\{\frac{\partial f}{\partial t}(x,t)\right\}=\psi(s)F(x,s)-\phi^{\alpha(s)}f(x,0). \tag{3.1}$$

Which completes the proof. □

Theorem 3.2. *Second Partial Derivative: If* $S\{f(x,t);s\}=F(x,s)$, *then*

$$S\left\{\frac{\partial^2 f}{\partial t^2}(x,t)\right\}=\psi^2(s)F(x,s)-\phi^{\alpha}(s)[\psi(s)f(x,0)-f'(x,0)].$$

Proof. Consider the function $S\{f(x,t);s\}=F(x,s)$, and after applying transform (2.1), considering $\gamma(t)=1$, we have

$$S\left\{\frac{\partial^2 f}{\partial t^2}(x,t)\right\}=\phi^{\alpha}(s)\int_0^{\infty}e^{-\psi(s)t}\frac{\partial^2 f}{\partial t^2}(x,t)dt.$$

Using integration by parts and (3.1), we get

$$S\left\{\frac{\partial^2 f}{\partial t^2}(x,t)\right\}=\phi^{\alpha}(s)\Big[\frac{\partial f}{\partial t}(x,t)e^{-\psi(s)t}\Big]_0^{\infty}+\phi^{\alpha}(s)\psi(s)\int_0^{\infty}\frac{\partial f}{\partial t}(x,t)e^{-\psi(s)}dt.$$
$$=\phi^{\alpha}(s)(0-f'(x,0))+\psi(s)[\psi(s)F(x,s)-\phi^{\alpha(s)}f(x,0)]dt.$$

This implies

$$S\left\{\frac{\partial^2 f}{\partial t^2}(x,t)\right\}=\psi^2(s)F(x,s)-\phi^{\alpha}(s)[\psi(s)f(x.0)+f'(x,0)].$$

Which completes the proof. □

Remark 3.1. Partial derivative of any order (with t as variable) can be deduced from this nth order partial derivative.

$$S\left\{\frac{\partial^n f}{\partial t^n}(x,t)\right\}=\psi^n(s)F(x,s)-\phi^{\alpha}(s)\sum_{k=0}^{n-1}\psi^{n-k-1}(s)f^k(0).$$

3.2 Transform on Partial derivatives with variable x

In this subsection, we apply the transform on Partial derivatives w.r.t variable x.

Theorem 3.3. *First Partial Derivative: If* $S\{f(x,t);s\} = F(x,s)$, *then* $S\left\{\frac{\partial f}{\partial x}(x,t)\right\} = \frac{d}{dx}F(x,s)$.

Proof. Consider the function $S\{f(x,t);s\} = F(x,s)$, and after applying transform (2.1), considering $\gamma(t) = 1$, we have

$$S\left\{\frac{\partial f}{\partial x}(x,t)\right\} = \phi^{\alpha}(s)\int_0^{\infty} e^{-\psi(s)t}\frac{\partial f}{\partial x}(x,t)dt.$$

Using transform (2.1) along with Leibniz rule results in

$$\begin{aligned} S\left\{\frac{\partial f}{\partial x}(x,t)\right\} &= \frac{\partial}{\partial x}\left\{\phi^{\alpha}(s)\int_0^{\infty} e^{-\psi(s)t}f(x,t)dt\right\}.S\left\{\frac{\partial f}{\partial x}(x,t)\right\}, \\ &= \frac{d}{dx}F(x,s). \end{aligned} \tag{3.2}$$

Which completes the proof. □

Theorem 3.4. *Second Partial Derivative: If* $S\{f(x,t);s\} = F(x,s)$, *then*
$S\left\{\frac{\partial^2 f}{\partial x^2}(x,t)\right\} = \frac{d^2}{dx^2}F(x,s).$

Proof. Consider the function $S\{f(x,t);s\} = F(x,s)$, and after applying transform (2.1), considering $\gamma(t) = 1$, we have

$$S\left\{\frac{\partial^2 f}{\partial x^2}(x,t)\right\} = \phi^{\alpha}(s)\int_0^{\infty} e^{-\psi(s)t}\frac{\partial^2 f}{\partial x^2}(x,t)dt.$$

Using transform (2.1) along with Leibniz rule results in

$$\begin{aligned} S\left\{\frac{\partial^2 f}{\partial x^2}(x,t)\right\} &= \frac{\partial^2}{\partial x^2}\left\{\phi^{\alpha}(s)\int_0^{\infty} e^{-\psi(s)t}f(x,t)dt\right\}.S\left\{\frac{\partial^2 f}{\partial x^2}(x,t)\right\}, \\ &= \frac{d^2}{dx^2}F(x,s). \end{aligned}$$

Which completes the proof. □

Remark 3.2. Partial derivative of any order can be deduced from this nth order partial derivative w.r.t variable x.

$$S\left\{\frac{\partial^n f}{\partial x^n}(x,t)\right\} = \frac{d^n}{dx^n}F(x,s).$$

Table 1: *Shows the results of some functions after applying the new transform.*

Function $f(x,t) = S^{-1}F(x,s)$	New integral transform $F(x,s) = S\{f(x,t);s\}$
1	$\frac{\phi^n(s)}{\psi(s)}$
c	$\frac{c\phi^n(s)}{\psi(s)}$
$kf(x,t)$	$kF(x,s)$
t	$\frac{\phi^n(s)}{\psi^2(s)}$
t^2	$2\frac{\phi^n(s)}{\psi^3(s)}$
t^n	$\Gamma(n+1)\frac{\phi^n(s)}{\psi^{n+1}(s)}$
e^{xt}	$\frac{\phi^n(s)}{\psi(s)-x}$
$\sin(t)$	$\frac{\phi^n(s)}{\psi^2(s)+1}$
$\cos(t)$	$\frac{\psi(s)\phi^n(s)}{\psi^2(s)+1}$
$\sin(xt)$	$\frac{x\phi^n(s)}{\psi^2(s)+x^2}$
$\cos(xt)$	$\frac{\psi(s)\phi^n(s)}{\psi^2(s)+x^2}$
$\sinh(t)$	$\frac{\phi^n(s)}{\psi^2(s)-1}$
$\cosh(t)$	$\frac{\psi(s)\phi^n(s)}{\psi^2(s)-1}$
$\sinh(xt)$	$\frac{x\phi^n(s)}{\psi^2(s)-x^2}$
$\cosh(xt)$	$\frac{\psi(s)\phi^n(s)}{\psi^2(s)-x^2}$
$e^{xt}\sin(xt)$	$\frac{x\phi^n(s)}{(\psi(s)-x)^2+x^2}$
$e^{xt}\cos(xt)$	$\frac{(\psi(s)-x)\phi^n(s)}{(\psi(s)-x)^2+x^2}$
$e^{xt}\sinh(xt)$	$\frac{x\phi^n(s)}{(\psi(s)-x)^2-x^2}$
$e^{xt}\cosh(xt)$	$\frac{(\psi(s)-x)\phi^n(s)}{(\psi(s)-x)^2-x^2}$
$\int_0^t f(x,t)dt$	$\frac{1}{\psi(s)}F(x,s)$
$\Sigma_{k=0}^{n} a_k t^k$	$\Sigma_{k=0}^{n} a_k \Gamma(n+1)\frac{\phi^n(s)}{\psi^{n+1}(s)}$
$tf(x,t)$	$\frac{-\phi^n(s)}{\psi'(s)}\left[\frac{F(x,s)}{\phi^n(s)}\right]'$
$t^2 f(x,t)$	$\frac{\phi^n(s)}{\psi'(s)}\cdot\left[\frac{1}{\psi'(s)}\left(\frac{F(x,s)}{\phi^n(s)}\right)'\right]'$
$t^n f(x,t)$	$(-1)^n\left(\frac{\phi^n(s)}{\psi'(s)}\right)\left[\frac{1}{\psi'(s)}\left(\frac{1}{\psi'(s)}\left(\cdots\left\{\frac{1}{\psi'(s)}\left(\frac{F(x,s)}{\phi^n(s)}\right)'\right\}'\right)'\cdots\right)\right]'$

4 Transform on some well-known Partial Differential Equations

In this section, the proposed most generalized fractional order integral transform is applied on some important well known partial differential equations of first and second order i.e., Heat, Wave, Laplace, and Telegraphers equation and their particular solution is provided.

Example 1. Consider the first order IVP, with an initial condition [34]

$$f_x(x,t) = 2f_t(x,t) + f(x,t), \qquad f(x,0) = 6e^{-3x}. \tag{4.1}$$

and f is bounded for $x > 0$, and $t > 0$. After applying transform (2.1), to (4.1), we get

$$S\{f_x(x,t)\} = S\{2f_t(x,t)\} + S\{f(x,t)\}. \tag{4.2}$$

After applying the properties of transform and ICs, (4.2) simplifies as,

$$\frac{d}{dx}F(x,s) - (2\psi(s)+1)F(x,s) = -12\phi^{\alpha}(s)e^{-3x}, \tag{4.3}$$

The (4.3) is required linear ODE. The integrating factor of the above equation is $e^{-(2\psi(s)+1)x}$.

The (4.3) takes the form

$$\frac{d}{dx}\left[e^{-(2\psi(s)+1)}F(x,s)\right] = -12\phi^{\alpha}(s)e^{-(2\psi(s)+4)x}. \tag{4.4}$$

After integration, the (4.4), gives,

$$\begin{aligned} e^{-(2\psi(s)+1)}F(x,s) &= -12\frac{\phi^{\alpha}(s)e^{-(2\psi(s)+4)x}}{-(2\psi(s)+4)} + c, \\ &= \frac{6\phi^{\alpha}(s)e^{-(2\psi(s)+4)x}}{e^{-(2\psi(s)+1)}(\psi(s)+2)} + \frac{c}{e^{-(2\psi(s)+1)}}, \\ &= \frac{6\phi^{\alpha}(s)e^{-3x}}{(\psi(s)+2)} + ce^{2\psi(s)+1}. \end{aligned}$$

As, f is bounded so c, should be zero, the above equation become

$$F(x,s) = \frac{6\phi^{\alpha}(s)e^{-3x}}{(\psi(s)+2)}. \tag{4.5}$$

After applying the inverse transform on (4.5), we get the solution

$$f(x,t) = 6e^{-3x}e^{-2t}.$$

The graph of the solution is presented in Figure 1.

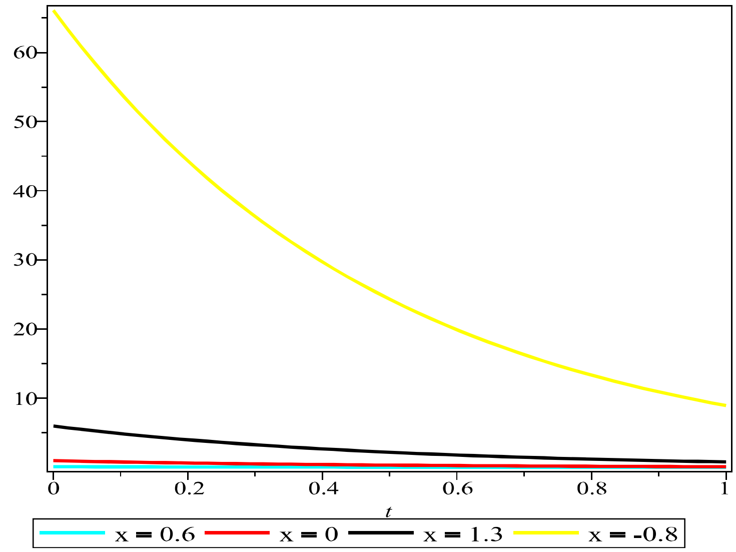

Figure 1: Graph of a linear partial differential equation for varying parameters when $0 \leq t \leq 1$.

Example 2. Consider the Laplace equation, with an initial condition [35]

$$f_{xx}(x,t)+f_{tt}(x,t)=0, \qquad f(x,0)=0, \quad \text{and} \quad f_t(x,0)=cos(x). \tag{4.6}$$

After applying the transform (2.1), on (4.6), we get

$$S\{f_{xx}(x,t)\}+S\{2f_{tt}(x,t)\}=S\{0\}. \tag{4.7}$$

After applying the properties of transform and ICs, (4.7) simplifies as,

$$\left[\frac{d^2}{dx^2}+\psi^2(s)\right]F(x,s) = \phi^\alpha(s)cos(x). \tag{4.8}$$

After substitution, the (4.8) becomes

$$F(x,s)=\frac{\phi^\alpha(s)cos(x)}{(D^2+\psi^2(s))}.$$

The solution of the above equation is

$$F(x,s)=\frac{\phi^\alpha(s)cos(x)}{(\psi^2(s)-1)}. \tag{4.9}$$

After applying the inverse transform on (4.9), the required solution is obtained. The graph of the solution is presented in Figure 2.

$$f(x,t) = \cos(x)\sinh(t).$$

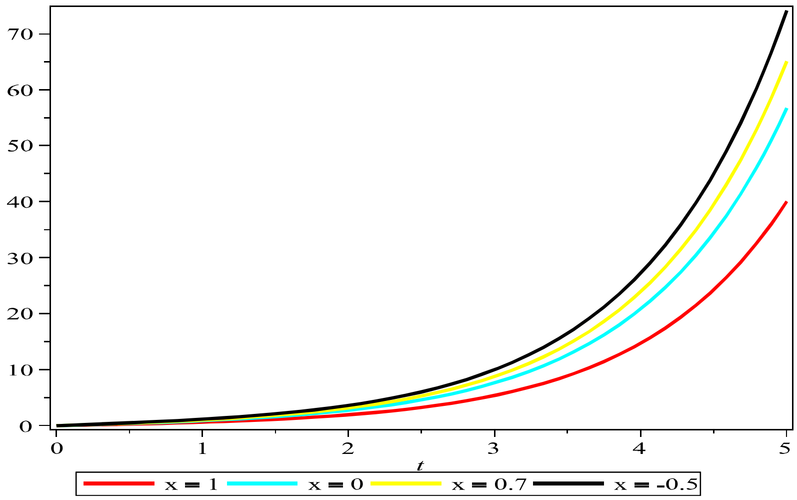

Figure 2: Graph of a Laplace partial differential equation for varying parameters when $0 \le t \le 1$.

Example 3. Consider the one-dimensional homogeneous Wave equation, with an initial condition [35]

$$f_{tt}(x,t) - 4f_{xx}(x,t) = 0, \quad x,t > 0, \quad f(x,0) = sin(\pi x), \quad f_t(x,0) = 0. \tag{4.10}$$

After applying transform (2.1), to (4.10), we get

$$S\{f_{tt}(x,t)\} - S\{4f_{xx}(x,t)\} = S\{0\}. \tag{4.11}$$

After applying the properties of transform and ICs, (4.11), simplifies as,

$$\left[\psi^2(s) - 4\frac{d^2}{dx^2}\right]F(x,s) - \phi^\alpha(s)\psi(s)sin(\pi x) = 0. \tag{4.12}$$

After substitution, the (4.12) becomes

$$F(x,s) = \frac{\phi^\alpha(s)\psi(s)sin(\pi x)}{(\psi^2(s) - 4D^2)}.$$

This implies

$$F(x,s) = \frac{\phi^\alpha(s)\psi(s)sin(\pi x)}{(\psi^2(s) + (2\pi)^2)}. \tag{4.13}$$

The required solution is obtained, after applying the inverse transform on (4.17). The graph of the solution is presented in Figure 3.

$$f(x,t) = sin(\pi x)cos2(\pi t).$$

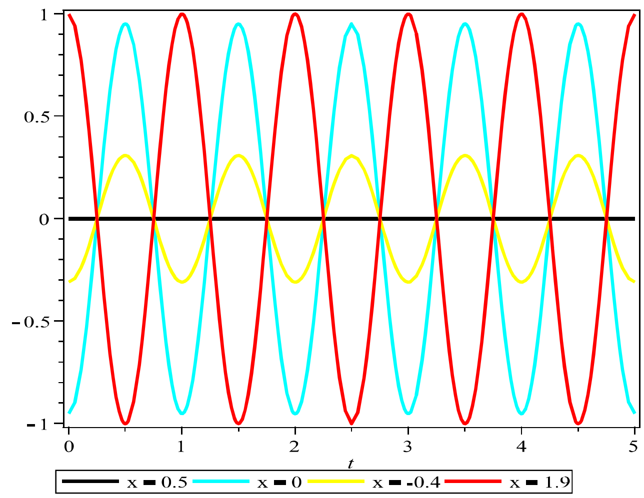

Figure 3: Graph of a wave partial differential equation representing the transverse vibration of a string for varying parameters when $0 \le t \le 1$ and wave speed is 4.

Example 4. Consider the homogeneous Heat equation in normalized form, with an initial condition [35]

$$4f_t(x,t) = f_{xx}(x,t), \quad x,t > 0, \quad f(x,0) = sin(\frac{\pi}{2}x), \quad f_t(x,0) = 0. \tag{4.14}$$

After applying transform (2.1), on (4.10), we get

$$S\{4f_t(x,t)\} = S\{f_{xx}(x,t)\}. \tag{4.15}$$

After applying the properties of transform and ICs, (4.19) simplifies as,

$$\left[\frac{d^2}{dx^2} - 4\psi(s)\right]F(x,s) = -4\phi^\alpha(s)sin\left(\frac{\pi}{2}x\right) \tag{4.16}$$

After substitution, the (4.20) becomes

$$F(x,s) = -4\frac{\phi^\alpha(s)sin\left(\frac{\pi}{2}x\right)}{(D^2 - 4\psi(s))}.$$

This implies

$$F(x,s) \; = \frac{\phi^{\alpha}(s) sin\left(\frac{\pi}{2}x\right)}{\left(\frac{\pi^2}{16}+\psi(s)\right)}. \tag{4.17}$$

The required solution is obtained by using the inverse transform on (4.17).

$$S^{-1}\{F(x,s)\} = sin\left(\frac{\pi}{2}x\right) S^{-1}\left\{\frac{\phi^{\alpha}(s)}{\left(\frac{\pi^2}{16}+\psi(s)\right)}\right\},$$

$$f(x,t) = sin\left(\frac{\pi}{2}x\right) e^{-\frac{\pi^2}{16}t}.$$

Hence which is required solution. The graph of the solution is presented in Figure 4.

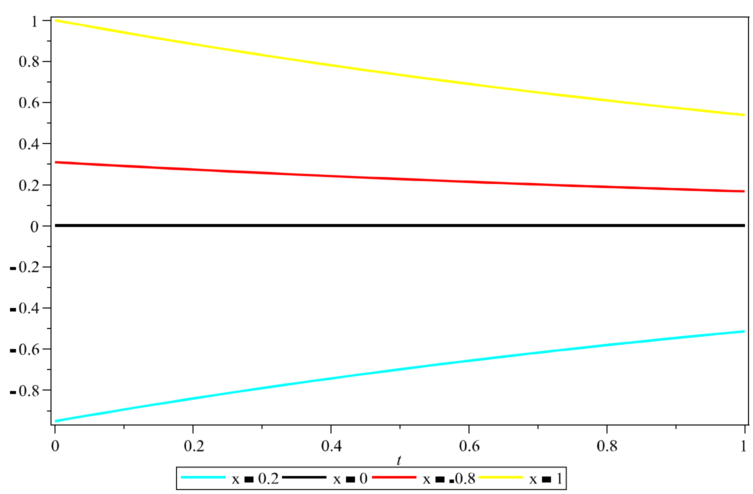

Figure 4: Graph of a 1D wave partial differential equation for varying parameters when $0 \leq t \leq 1$.

Example 5. Consider the Telegraphers equation, with an initial condition [35]

$$f_{xx}(x,t) = f_{tt}(x,t) + 2f_t + f, \qquad 0 < x < 1, t > 0$$
$$f(x,0) = sin\left(\frac{\pi}{2}x\right), \quad \text{and} \quad f_t(x,0) = 0. \tag{4.18}$$

After applying transform (2.1), to (4.18), we get

$$S\{f_{xx}(x,t)\} = S\{f_{tt}(x,t)\} + S\{2f_t(x,t)\} + S\{f(x,t)\}. \tag{4.19}$$

After applying the properties of transform and ICs, (4.19) simplifies as,

$$\frac{d^2}{dx^2}F(x,s) - (\psi^2(s) + 2\psi(s) + 1)F(x,s) = -\phi^\alpha(s)\psi(s)e^x,$$
$$\left[\frac{d^2}{dx^2} - (\psi^2(s) + 2\psi(s) + 1)\right]F(x,s) = -\phi^\alpha(s)\psi(s)e^x. \tag{4.20}$$

After substitution, the (4.20) becomes

$$F(x,s) = -\frac{-\phi^\alpha(s)\psi(s)e^x}{\left[D^2 - (\psi^2(s) + 2\psi(s) + 1)\right]}.$$

This implies

$$F(x,s) = \frac{\phi^\alpha(s)e^x}{(\psi(s)+2)}. \tag{4.21}$$

By using the inverse transform on equation (4.21), we get the solution

$$f(x,t) = e^x e^{-2t}.$$

The graph of the solution is presented in Figure 5.

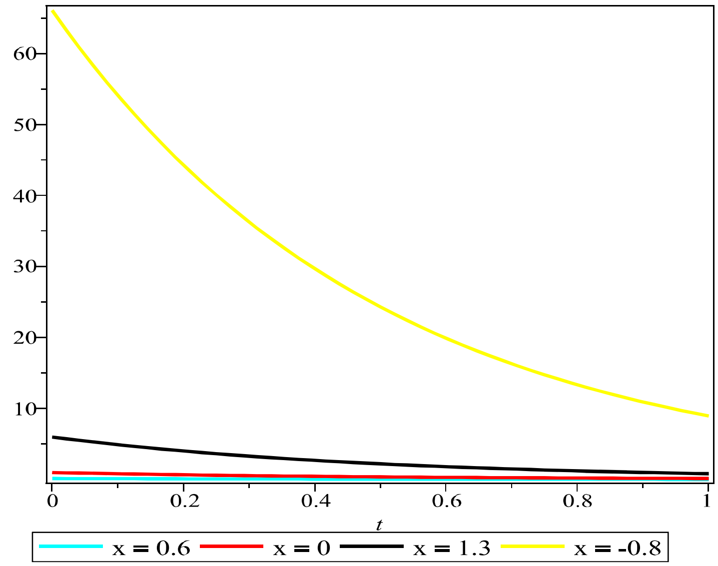

Figure 5: Graph of a Telegraphers partial differential equation for varying parameters when $0 \leq t \leq 1$.

5 Conclusions

In this paper, we developed a novel, most generalized fractional integral transform, capable of solving a wide range of integral and differential equations of both integer and fractional order. Furthermore, convergence and some other important properties of the transforms have been proved, and results are depicted in Table 1. Moreover, we calculated the partial derivatives of different orders with variable t and x. Our main work encompasses particular solutions to various well-known linear first and second-order partial differential equations, such as the Laplace, Wave, Homogeneous one-dimensional Heat in normalized form, and Telegraphers equations.

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