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GENERALISATION OF HERMITE-HADAMARD LIKE INEQUALITIES FOR $(\alpha,\beta,\gamma,\mu)-$CONVEX FUNCTIONS IN MIXED KIND WITH APPLICATIONS

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Abstract. *In the current paper, few generalised inequalities of Hermite-Hadamard like for functions whose modulus of the derivatives are $(\alpha,\beta,\gamma,\mu)-$convex in mixed kind and applications for theory of Probability & Numerical integration are deduced. Several established consequences of several published will capture as especial cases. Moreover, we deduce more especial cases of the class of $(\alpha,\beta,\gamma,\mu)-$convex function on various choices of α,β,γ,μ.*

1. **Introduction and Definition**

About the features of convex functions, we code some lines from [16] ‘Numerous problems in applied and pure mathematics involve convex functions. They act extremely crucial role in research of problems of non-linear and linear programming. The convex functions’ theory falls under the broader topic of convexity. However, this theory significantly affects practically every area of mathematical sciences. One of the earliest areas of mathematics where the concept of convexity is necessary for graphic analysis. Calculus provides us with a useful technique, the second derivative test, to identify convexity’.

We must generalise the idea of convex functions in order to generalise Ostrowski's inequality. In this way, we may quickly identify the generalisations and specific instances of the inequality. We recollect several definitions from the literature [2] for various convex functions.

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Definition 1.1. A function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as convex, if

$$g(\tau y + (1-\tau)z) \leq \tau g(y) + (1-\tau)g(z), \tag{1.1}$$

$\forall\, z, y \in K,\ \tau \in [0,1]$.

We remind term of $P-$convex function (see [4]).

Definition 1.2. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as $P-$convex, if

$$g(\tau y + (1-\tau)z) \leq g(y) + g(z), \quad g \geq 0 \tag{1.2}$$

$\forall\, z, y \in K,\ \tau \in [0,1]$.

We remind definition of quasi-convex function from [7].

Definition 1.3. A function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex, if

$$g(\tau y + (1-\tau)z) \leq \max\{\, g(z), g(y)\, \} \tag{1.3}$$

$\forall\, y, z \in K,\ \tau \in [0,1]$.

Now we present term of s$-$convex function (see [15]).

Definition 1.4. Suppose $s \in (0,1]$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $s-$convex in the 1st kind, if

$$g(\tau y + (1-\tau)z) \leq \tau^s g(y) + (1-\tau^s)g(z), \tag{1.4}$$

$\forall\, y, z \in K,\ \tau \in [0,1]$.

Remark 1.5. If we include $s=0$ in the above then obtain the term of quasi-convex.

For second kind convexitY we recall term (see [15]).

Definition 1.6. Let $s \in (0,1]$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $s-$convex in the 2nd kind, if

$$g(\tau y + (1-\tau)z) \leq \tau^s g(y) + (1-\tau)^s g(z), \tag{1.5}$$

$\forall\, y, z \in K,\ \tau \in [0,1]$.

Remark 1.7. If we include $s = 0$ in the above then obtain the term of $P-$convex.

Definition 1.8. [8] Let $(s,r) \in [0,1]^2$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $(s,r)-$convex in the mixed kind, if

$$g(\tau y + (1-\tau)z) \leq \tau^{rs} g(y) + (1-\tau^r)^s g(z), \tag{1.6}$$

$\forall\, z, y \in K,\ \tau \in [0,1]$.

Definition 1.9. [6] Suppose $(\alpha,\beta) \in [0,1]^2$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex in the 1st kind, if

$$g(\tau y + (1-\tau)z) \leq \tau^\alpha g(y) + (1-\tau^\beta)g(z), \tag{1.7}$$

$\forall\, z, y \in K,\ \tau \in [0,1]$.

Definition 1.10. [6] Let $(\alpha,\beta) \in [0,1]^2$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex in the 2nd kind, if

$$g(\tau y + (1-\tau)z) \leq \tau^{\alpha} g(y) + (1-\tau)^{\beta} g(z), \tag{1.8}$$

$\forall\, z, y \in K,\ \tau \in [0,1]$.

Now the next important definition of $(\alpha,\beta,\gamma,\mu)-$convex function which is sequentially used in this article and it is extracted from [8].

Definition 1.11. Let $(\alpha,\beta,\gamma,\mu) \in [0,1]^4$. A function $g : K \subseteq [0,\infty) \rightarrow [0,\infty)$ is known as $(\alpha,\beta,\gamma,\mu)-$convex in the mixed kind, if

$$g(\tau y + (1-\tau)z) \leq (1-\tau^{\beta})^{\mu} g(z) + \tau^{\alpha\gamma} g(y), \tag{1.9}$$

$\forall\, y, z \in K,\ \tau \in [0,1]$.

Remark 1.12. The following scenarios are found in Definition 1.11 as especial cases.

(i) The $(\alpha,\beta)-$convex function in 1st kind is obtained if choose $\gamma = \mu = 1$ & $\alpha,\beta \in [0,1]$ in (1.9).
(ii) The $(\alpha,\mu)-$convex function in 2nd kind is obtained if choose $\beta = \gamma = 1$ & $\alpha,\mu \in [0,1]$ in (1.9).
(iii) The $(s,r)-$convex function in mixed kind is obtained if choose $\alpha = \mu = s$, $\beta = 1$, $\gamma = r$, here $s,r \in (0,1]$ in (1.9).
(iv) The $s-$convex function in 1st kind is obtained if choose $\alpha = \beta = s$ and $\gamma = \mu = 1$, here $s \in (0,1]$ in (1.9).
(v) The quasi$-$convex function is obtained if choose $\alpha = \beta = 0$, & $\gamma = \mu = 1$ in (1.9).
(vi) The $s-$convex function in 2nd kind is obtained if choose $\alpha = \mu = s$, $\beta = \gamma = 1$, here $s \in (0,1]$ in (1.9).
(vii) The $P-$convex function is obtained if choose $\alpha = \mu = 0$ & $\beta = \gamma = 1$ in (1.9).
(viii) The ordinary convex function is obtained if choose $\alpha = \beta = \gamma = \mu = 1$ in (1.9).

In practically all scientific fields, inequalities play a major impact. Our primary target is on Hermite–hadamard like inequalities, despite the discipline's enormous scope.

The convexity theory is closely connected to the inequalities theory. Many well-known Inequalities in literature consequence directly from applying convex functions. The Hermite-hadamard equation is a notable equation for convexity that has been thoroughly researched in recent decades. It is discovered independently by Hadamard & Hermite and it is stated as follows: Any Function $g : K \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be convex, here $k,j \in K$ along $j<k$, If & only if

$$g\left(\frac{j+k}{2}\right) \leq \frac{1}{k-j}\int_j^k g(y)dy \leq \frac{g(j)+g(k)}{2}, \tag{1.10}$$

this is known as inequality of Hermite-Hadamard. Eq. (1.10) has become a crucial pillar in the area of probability & optimization. Additionally, numerous researchers have refined or generalised equation (1.10) for convex, s-convex, & various other varieties of functions. Regarding the history of this inequality, we must see [14].

In [5], the following consequence had derived by Agarwal & Dragomir, which includes the Hermite-Hadamard like integral equation.

Proposition 1.13. *Let* $g : K^o \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mappinG in interior* K^o *of K, here* $k,j \in K^o$ *along* $k > j$. *If* $|g'|$ *is convex in interval* $[j,k]$. *Then the below equation holds*

$$\left|\frac{g(j)+g(k)}{2} - \frac{1}{k-j}\int_j^k g(u)du\right| \leq \frac{(k-j)(|g\prime(j)|+|g\prime(k)|)}{8}. \qquad (1.11)$$

For additional current consequences on Hermite-Hadamard like equations involving various classes of convexity, refer to [1, 3, 9, 10, 12, 14, 19].

In 2011, Kavurmaci et. al. [9] established few equations of Hermite-Hadamard like for convexity & applications by utilizing Hölder inequality & Powermean inequality. In 2016, Liu et. al. [12] established few novel equations of Hermite-Hadamard like for $MT-$convexity via classical integrals & Riemann-Liouville fractional integrals, respectively. In 2023, Mehmood et. al. [13] established few novel generalised inequalities of Hermite-Hadamard like for $(s,r)-$convex functions & applications by Power-mean & Hölder inequality.

The primary goal of the article is to generalise few Hermite-Hadamard like inequalities to $(\alpha,\beta,\gamma,\mu)-$convexity in mixed kind via classical integrals by employing the Hölder & Powermean inequalities. The applications also encompass areas such as probability & numerical integration. We will capture some findings of various articles [5, 9, 13] & also examine especial cases of class of $(\alpha,\beta,\gamma,\mu)-$convex function on different choices of α,β,γ,μ as remarks.

2. Generalisation of Hermite-Hadamard Like Inequalities

Regarding proof of primary findings, below Lemma (see [9]) is required.

Lemma 2.1. *Let* $g : K \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping in interval* $K^o \subset \mathbb{R}$ *here* $k,j \in K$ *along* $j<k$. *If* $g' \in L[j,k]$, *then*

$$\frac{(k-y)g(k)+(y-j)g(j)}{k-j} - \frac{1}{k-j}\int_j^k g(u)du$$

$$= \frac{(y-j)^2}{k-j}\int_o^1 (1-\tau)g'(\tau y+(1-\tau)j)d\tau + \frac{(k-y)^2}{k-j}\int_o^1 (1-\tau)g'(\tau y+(1-\tau)k)d\tau.$$

The below consequences may be derived by employing Lemma 2.1

Theorem 2.2. *Let* $g : K \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping in interval* $K^o \subset \mathbb{R}$ ε $g' \epsilon L[j,k]$, *here* $k,j \in K$ *along* $k>j$. *If* $|g'|$ *is* $(\alpha,\beta,\gamma,\mu)-$ *convex in [j,k], then*

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j} - \frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{(y-j)^2}{k-j}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{|g'(j)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+ \frac{(k-y)^2}{k-j}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{|g'(k)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

for every $g \in [j,k]$ & $\beta > 0$.

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Proof. Utilizing Lemma 2.1 & taking Modulus, then

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{(y-j)^2}{k-j}\int_o^1(1-\tau)|g'(\tau y+(1-\tau)j|d\tau+\frac{(k-y)^2}{k-j}\int_o^1(1-\tau)|g'(\tau y+(1-\tau)k)|d\tau$$

Since $|g'|$ is $(\alpha,\beta,\gamma,\mu)$−convex in mixed kind, then

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{(y-j)^2}{k-j}\int_o^1(1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|+\left(1-\tau^{\beta}\right)^{\mu}g'(j)\right]d\tau$$
$$+\frac{(k-y)^2}{k-j}\int_o^1(1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|+\left(1-\tau^{\beta}\right)^{\mu}|g'(k)|\right]d\tau$$
$$=\frac{(y-j)^2}{k-j}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|g'(j)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\frac{(k-y)^2}{k-j}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|g'(k)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

thus finished the proof.

Note: Where B is Beta function, symbolically described as $B(l,m)=\int_o^1\tau^{l-1}(1-\tau)^{m-1}d\tau=\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$. Since $\Gamma(l)=\int_o^{\infty}e^{-u}u^{l-1}du.$

Remark 2.3. The following scenarios are found in Theorem 2.2 as especial cases.

(i) The (α,β)−convex function in 1st kind is obtained if choose $\gamma=\mu=1$ & α , $\beta\in[0,1]$ in Theorem 2.2.
(ii) The (α,μ)−convex function in 2nd kind is obtained if choose $\beta=\gamma=1$ & α , $\mu\in[0,1]$ in Theorem 2.2.
(iii) The s−convex function in 1st kind is obtained if choose $\alpha=\beta=s$ & $\gamma=\mu=1$, here $s\in(0,1]$ in Theorem 2.2.
(iv) The quasi−convex function is obtained if choose $\alpha=\beta=0$ & $\gamma=\mu=1$ in Theorem 2.2.
(v) The s−convex function in 2nd kind is obtained if choose $\alpha=\mu=s$, $\beta=\gamma=1$, here $s\in(0,1]$ in Theorem 2.2.
(vi) The P−convex function is obtained if we choose $\alpha=\mu=0$ and $\beta=\gamma=1$ in Theorem 2.2.

Remark 2.4. We attain the Theorem 2.2 of [13] if choose $\alpha=\mu=s$, $\beta=1$, $\gamma=r$, where s , $r\in(0,1]$ in Theorem 2.2.

Remark 2.5. We attain the Theorem 4 of [9] if choose $\alpha=\beta=\gamma=\mu=1$ in Theorem 2.2.

Corollary 2.6. *In Theorem* 2.2*, choosing* $y=\frac{j+k}{2}$ *we get*

$$\left|\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{k-j}{4}\left[\frac{2\left|g'\left(\frac{j+k}{2}\right)\right|}{(\alpha\gamma+1)(\alpha\gamma+2)}+(|g'(j)|+s|g'(k)|)\left(\frac{B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)}{\beta}\right)\right]$$

Remark 2.7. Few Remarks regarding Corollary 2.6 are below as especial cases.

(i) By utilizing the convexity property of $|g'|$ in Corollary 2.6, we get established equation (1.11) (capture Theorem 2.2 of article [5]).
(ii) The (α,β)−convex function in the 1st kind is obtained if we choose $\gamma = \mu = 1$ & α , $\beta \in [0,1]$ in Corollary 2.6.
(iii) The (α,μ)−convex function in the 2nd kind is obtained if we choose $\beta = \gamma = 1$ & α , $\mu \in [0,1]$ in Corollary 2.6.
(iv) The s−convex function in the 1st kind is obtained if we choose $\alpha = \beta = s$ & $\gamma = \mu = 1$ where $s \in (0,1]$ in Corollary 2.6.
(v) The quasi−convex function is obtained if we choose $\alpha = \beta = 0$ & $\gamma = \mu = 1$ in Corollary 2.6.
(vi) The s−convex function in the 2nd kind is obtained if we choose $\alpha = \mu = s$, $\beta = \gamma = 1$ where $s \in (0,1]$ in Corollary 2.6.
(vii) The P−convex function is obtained if we choose $\alpha = \mu = 0$ and $\beta = \gamma = 1$ in Corollary 2.6.

Remark 2.8. If choose $\alpha = \mu = s$, $\beta = 1$, $\gamma = r$, where s , $r \in (0,1]$ in Corollary 2.6, we attain the Corollary 2.5 of [13].

Remark 2.9. If we choose $\alpha = \beta = \gamma = \mu = 1$ in Corollary 2.6, we attain Corollary 2 of [9].

Theorem 2.10. *Let $g : K \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping in interval $K^o \subset \mathbb{R}$ ε $g' \epsilon L[j,k]$, here $k,j \in K$ along $j<k$. If $|g'|^{\frac{p}{p-1}}$ is $(\alpha,\beta,\gamma,\mu)$−convex in interval $[j,k]$ & for few fixed $1 < q$, then*

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(y-j)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)}+\frac{|g'(j)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+(k-y)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)}+\frac{|g'(k)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right]$$

for every $g \in [j,k]$ & $\beta > 0$.

Proof. Utilizing Lemma 2.1 & $(\alpha,\beta,\gamma,\mu)$−convexity of $|g'|$ & then implementing the widely recognized Hölder inequality, we get

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{(y-j)^2}{k-j}\int_o^1(1-\tau)|g'(\tau y+(1-\tau)j|d\tau+\frac{(k-y)^2}{k-j}\int_o^1(1-\tau)|g'(\tau y+(1-\tau)k)|d\tau$$

$$\leq \frac{(y-j)^2}{k-j}\int_o^1(1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|+\left(1-\tau^{\beta}\right)^{\mu}|g'(j)|\right]d\tau$$
$$+\frac{(k-y)^2}{k-j}\int_o^1(1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|+\left(1-\tau^{\beta}\right)^{\mu}|g'(k)|\right]d\tau$$
$$\leq \frac{(y-j)^2}{k-j}\left(\int_o^1(1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_o^1\left(\tau^{\alpha\gamma}|g'(y)|+\left(1-\tau^{\beta}\right)^{\mu}|g'(j)|\right)^q d\tau\right]^{\frac{1}{p}}$$
$$+\frac{(k-y)^2}{k-j}\left(\int_o^1(1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_o^1\left(\tau^{\alpha\gamma}|g'(y)|+\left(1-\tau^{\beta}\right)^{\mu}|g'(k)|\right)^q d\tau\right]^{\frac{1}{p}}$$

$$\leq \frac{(y-j)^2}{k-j}\left(\int_o^1(1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_o^1\tau^{\alpha\gamma}|g'(y)|^q d\tau+\int_o^1\left(1-\tau^{\beta}\right)^{\mu}|g'(j)|^q d\tau\right]^{\frac{1}{p}}$$
$$+\frac{(k-y)^2}{k-j}\left(\int_o^1(1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_o^1\tau^{\alpha\gamma}|g'(y)|^q d\tau+\int_o^1\left(1-\tau^{\beta}\right)^{\mu}|g'(k)|^q d\tau\right]^{\frac{1}{p}}$$
$$\leq \frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(y-j)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)}+\frac{|g'(j)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}+(k-y)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)}+\frac{|g'(k)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right]$$

Remark 2.11. Since we have provided remarks (i) through (vi) for Theorem 2.2, all of the remarks employ to Theorem 2.10.

Remark 2.12. We attain the Theorem 2.8 of [13] if choose $\alpha=\mu=s$, $\beta=1$, $\gamma=r$, here $s,r\in(0,1]$ in Theorem 2.10.

Remark 2.13. We attain Theorem 5 of [9] if choose $\alpha=\beta=\gamma=\mu=1$ in Theorem 2.10.

Corollary 2.14. *In Theorem* 2.10, *choosing* $y=\frac{j+k}{2}$ *we get*

$$\left|\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{k-j}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|g'(j)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)+\frac{\left|g'\left(\frac{j+k}{2}\right)\right|^q}{\alpha\gamma+1}\right)^{\frac{1}{q}}+\left(\frac{|g'(k)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)+\frac{\left|g'\left(\frac{j+k}{2}\right)\right|^q}{\alpha\gamma+1}\right)^{\frac{1}{q}}\right]$$

Remark 2.15. Since we have provided remarks (ii) through (vii) for Corollary 2.6, all of the remarks employ to Corollary 2.14.

Remark 2.16. We capture the Corollary 2.11 of [13] if choose $\alpha = \mu = s$, $\beta = 1, \gamma = r$, here $s,r \in (0,1]$ in Corollary 2.14,.

Remark 2.17. We capture the Corollary 3 of [9] if choose $\alpha = \beta = \gamma = \mu = 1$ in Corollary 2.14.

Theorem 2.18. *Let $g : K \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $K^{o} \subset \mathbb{R}$ ε $g' \epsilon L[j,k]$, here $j,k \in K$ along $j<k$. If $|g'|^q$ is $(\alpha,\beta,\gamma,\mu)-$convex on $[j,k]$ & for few fixed $1 \leq q$, then*

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j} - \frac{1}{k-j}\int_j^k g(u)du\right|$$

$$\leq \frac{1}{2^{1-\frac{1}{q}}(k-j)}\left[(y-j)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{|g'(j)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right)^{\frac{1}{q}}\right.$$

$$\left. +(k-y)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{|g'(k)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right)^{\frac{1}{q}}\right]$$

for every $g \in [j,k]$, $q = \frac{p}{p-1}$ & $\beta > o$.

Proof. Consider $1 \leq q$ & Utilizing Lemma 2.1 & implementing the widely recognized power-mean inequality, we get

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j} - \frac{1}{k-j}\int_j^k g(u)du\right|$$

$$\leq \frac{(y-j)^2}{k-j}\int_o^1 (1-\tau)|g'(\tau y+(1-\tau)j)|d\tau + \frac{(k-y)^2}{k-j}\int_o^1 (1-\tau)|g'(\tau y+(1-\tau)k|d\tau$$

$$\leq \frac{(y-j)^2}{k-j}\left(\int_o^1 (1-\tau)d\tau\right)^{1-\frac{1}{q}}\left(\int_o^1 (1-\tau)|g'(\tau y+(1-\tau)j|^q d\tau\right)^{\frac{1}{q}}$$

$$+\frac{(k-y)^2}{k-j}\left(\int_o^1 (1-\tau)d\tau\right)^{1-\frac{1}{q}}\left(\int_o^1 (1-\tau)|g'(\tau y+(1-\tau)k|^q d\tau\right)^{\frac{1}{q}}$$

Since $|g'|$ is $(\alpha,\beta,\gamma,\mu)-$convex in mixed kind & first taking term

$$\int_o^1 (1-\tau)|g'(\tau y+(1-\tau)j|^q d\tau$$

$$\leq \int_o^1 (1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|^q + \left(1-\tau^{\beta}\right)^{\mu}|g'(j)|^q\right]d\tau$$

$$= \frac{|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{|g'(j)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)$$

Analogously,

$$\int_o^1 (1-\tau)|g'(\tau y+(1-\tau)k|^q d\tau \leq \frac{|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{|g'(k)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)$$

We obtain the desired result by combining all above inequalities.

Remark 2.19. Since we have provided remarks (i) through (vi) for Theorem 2.2, all of the remarks employ to Theorem 2.18.

Remark 2.2o. We attain the Theorem 2.14 of [13] if choose $\alpha = \mu = s$, $\beta = 1$, $\gamma = r$, where $s,r \in (0,1]$ in Theorem 2.18.

Remark 2.21. We attain the Theorem 7 of [9] if choose $\alpha = \beta = \gamma = \mu = 1$ in Theorem 2.18.

Corollary 2.22. *In Theorem* 2.18*, choosing* $y = \frac{j+k}{2}$ *we get*

$$\left|\frac{g(j)+g(k)}{2} - \frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq 2^{\frac{1}{q}-3}(k-j)\left[\left(\frac{\left|g'\left(\frac{j+k}{2}\right)\right|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{|g'(j)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right)^{1/q}\right.$$
$$\left.+\left(\frac{\left|g'\left(\frac{j+k}{2}\right)\right|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{|g'(k)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right)^{1/q}\right]$$

Remark 2.23. Since we have provided remarks (ii) through (vii) for Corollary 2.6, all of the remarks employ to Corollary 2.22.

Remark 2.24. We capture the Corollary 2.17 of [13] if choose $\alpha = \mu = s$, $\beta = 1$,$\gamma = r$, here $s,r \in (0,1]$ in Corollary 2.22,.

Remark 2.25. We capture the Corollary 4 of [9] if choose $\alpha = \beta = \gamma = \mu = 1$ in Corollary 2.22.

3. Application to Numerical Integration

3.1. **The Trapezoidal Formula.** Let $d : j = \theta_0 < \theta_1 < \cdots < \theta_n = k$ be division of interval $[j,k]$, here $h_i = \theta_{i+1} - \theta_i$, $(i = 0,1,2,\cdots,n-1)$ & considering the quadrature formula

$$\int_j^k g(u)du = Q(g,d) + R(g,d), \tag{3.1}$$

where

$$Q(g,d) = \sum_{i=o}^{n-1}((\theta_{i+1} - y)g(\theta_{i+1}) + (y - \theta_i)g(\theta_i))$$

the related approximation error is represented by $R(g,d)$ for the trapezoidal variant.

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Theorem 3.1. *With all the suppositions of Theorem* 2.2, *for every division* d *of the interval* $[j,k]$. *Then, the trapezoidal error estimate in equation (3.1) satisfies*

$$|R(g,d)| \leq \sum_{i=o}^{n-1}(y-\theta_i)^2\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|g'(\theta_i)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\sum_{i=o}^{n-1}(\theta_{i+1}-y)^2\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|g'(\theta_{i+1})|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

for every $g \in [j,k]$ & $\beta > o$.

Proof. Utilizing Theorem 2.2 on sub-interval $[\theta_i, \theta_{i+1}]$, here $i = 0,1,2,\dots, n-1$, we get

$$\left|\frac{(\theta_{i+1}-y)g(\theta_{i+1})+(y-\theta_i)g(\theta_i)}{h_i}-\frac{1}{h_i}\int_{\theta_i}^{\theta_{i+1}}g(u)du\right|$$
$$\leq \frac{(y-\theta_i)^2}{h_i}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|g'(\theta_i)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\frac{(\theta_{i+1}-y)^2}{h_i}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|g'(\theta_{i+1})|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

Now we take summation over i in equation (3.1) from 0 to $n-1$, then

$$\left|\int_j^k g(u)du-Q(g,d)\right|=\left|\sum_{i=o}^{n-1}\left[\int_{\theta_i}^{\theta_{i+1}}g(u)du-((\theta_{i+1}-y)g(\theta_{i+1})+(y-\theta_i)g(\theta_i))\right]\right|$$
$$\leq \sum_{i=o}^{n-1}\left|\int_{\theta_i}^{\theta_{i+1}}g(u)du-\left((\theta_{i+1}-y)g(\theta_{i+1})+(y-\theta_i)g(\theta_i)\right)\right|$$
$$\leq \sum_{i=o}^{n-1}(y-\theta_i)^2\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|g'(\theta_i)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\sum_{i=o}^{n-1}(\theta_{i+1}-y)^2\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|g'(\theta_{i+1})|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

Which finished the proof.

Remark 3.2. We can also derive above similar findings for Corollary 2.6, Theorem 2.10, Corollary 2.14, Theorem 2.18 & Corollary 2.22.

Remark 3.3. The whole 2nd section's remarks are likewise applicable to Remarks 3.2 & Theorem 3.1.

4. **Applications to Probability Theory**

Assume Y is a random variable taking values in the finite $[j,k]$ along probability density function $g : [j,k] \to [0,1]$ & cumulative distribution function $H(y) = P(Y \leq y) = \int_j^k g(u)du$.

Theorem 4.1. *With all the suppositions of Theorem* 2.2, *then*

$$\left|\frac{(k-y)H(k)+(y-j)H(j)}{k-j}-\frac{k-E(Y)}{k-j}\right|$$

$$\leq \frac{(y-j)^2}{k-j}\left[\frac{|H'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|H'(j)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

$$+\ \frac{(k-y)^2}{k-j}\left[\frac{|H\prime(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{|H\prime(k)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right] \quad (4.1)$$

for every $g \in [j,k]$ *&* $\beta > 0$. *Here* $E(Y)$ *is expectation of* Y.

Proof. Choose $g = H$, we acquire (4.1), by implementing the below equation in

Theorem 2.2.

$$E(Y) = \int_j^k uH(u)du = k - \int_j^k H(u)du$$

$\because H(j) = 0$ & $H(k) = 1$.

Remark 4.3. The whole 2nd section's remarks are likewise applicable to Remarks 4.2 & Theorem 4.1.

5. **Conclusion**

In the current paper, we have generalised few findings regarding notoriety Hermite-Hadamard like inequalities for $(\alpha,\beta,\gamma,\mu)-$convex functions in the mixed kind via classical integrals by implementing the widely recognized power-mean & Hölder's inequalities & applications for theory of Probability & numerical integration are also deduced. We captured various findings of several published articles [5, 9, 13] & further deduced few especial cases of class of $(\alpha,\beta,\gamma,\mu)-$convexity on various choices of α,β,γ,μ as remarks.

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