LGJDX
ISSN No: 1008-0562
Journal Of Liaoning Technical University
Natural Science Edition

GENERALISATION OF HERMITE-HADAMARD TYPE INEQUALITIES FOR $(m, \alpha, \beta, \gamma, \mu)-$ CONVEX FUNCTIONS IN MIXED KINDS AND APPLICATIONS IN NUMERICAL INTEGRATION AND PROBABILITY THEORY

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Abstract: *In the present article, the generalised notion of (m, α, β, γ, μ) −convex(concave) function in mixed kind is introduced 1st time, which is the generalisation of 15. Our fundamental objective is to develop generalised inequalities of Hermite-Hadamard type for functions whose modulus of the derivatives are (m,α,β,γ,μ)−convex by implementing several techniques including Hölder's & power mean inequalities & applications for the theory of Probability & numerical integration are deduced. Moreover, we deduce more special cases of the class of (m,α,β,γ,μ)−convex function on various choices of m,α,β,γ,μ. Various established consequences of several published articles would be captured as special cases.*

Keywords: *Convex function, Hermite-Hadamard inequality, Power-mean inequality, H ö lder inequality, Beta function, Gamma function, Numerical integration, and Probability density function.*

1. **Introduction and Definition**

About the features of convex functions, we code some lines from [20] ‘Numerous problems in applied and pure mathematics involve convex functions. They play an extremely crucial role in the research of problems of non-linear and linear programming. The convex functions’ theory falls under the broader topic of convexity. However, this theory significantly affects practically every area of mathematical sciences. One of the earliest areas of mathematics where the concept of convexity is necessary for graphic analysis. Calculus provides us with a useful technique, the second derivative test, to identify convexity’.

We must generalise the idea of convex functions in order to generalise Ostrowski's inequality. In this way, we may quickly identify the generalisations and specific instances of the inequality. We recollect several definitions from the literature [2] for various convex functions.

Definition 1.1. A function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as convex(concave), if

$$g(\tau y + (1-\tau)z) \leq (\geq) \tau g(y) + (1-\tau)g(z), \qquad (1.1)$$

$\forall\, z, y \in K,\ \tau \in [0,1]$.

We remind term of $P-$convex function (see [4]).

Definition 1.2. A function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as $P-$convex(concave), if

$$g(\tau y + (1-\tau)z) \leq g(y) + g(z), \quad g \geq 0 \qquad (1.2)$$

$\forall\, z, y \in K,\ \tau \in [0,1]$.

We remind definition of quasi-convex function from [7].

Definition 1.3. A function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex(concave), if

$$g(\tau y + (1-\tau)z) \leq \max\{\, g(z), g(y) \,\} \qquad (1.3)$$

$\forall\, y, z \in K,\ \tau \in [0,1]$.

Now we present term of s−convex function (see [15]).

Definition 1.4. Suppose $s \in (0,1]$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $s-$convex(concave) in the 1st kind, if

$$g(\tau y + (1-\tau)z) \leq \tau^s g(y) + (1-\tau^s)g(z), \qquad (1.4)$$

$\forall\, y, z \in K,\ \tau \in [0,1]$.

Remark 1.5. If we include s=0 in the above then obtain the term of quasi-convex.

For second kind convexitY we recall term (see [19]).

Definition 1.6. Let $s \in (0,1]$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $s-$convex(concave) in the 2nd kind, if

$$g(\tau y + (1-\tau)z) \leq \tau^s g(y) + (1-\tau)^s g(z), \qquad (1.5)$$

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$\forall\ z, y \in K,\ \tau \in [0,1]$.

Remark 1.7. If we include $s = 0$ in the above then obtain the term of $P-$convex.

Now we present term of $m-$convex(concave) functions (see [9]).

Definition 1.8. Let $m \in [0,1]$. A function $g : [0,\infty) \to \mathbb{R}$ is known as $m-$convex (concave), if

$$g(\tau y + m(1-\tau)z) \leq (\geq) \tau g(y) + m(1-\tau)g(z), \tag{1.6}$$

$\forall\ y,z \in [0,\infty),\ \tau \in [0,1]$.

Remark 1.9. The terms of standard convex(concave) functions & star-shaped functions are obtained if choose $m = 1$ & $m = 0$ in the above inequality respectively.

Mihesan stated $(\alpha,m)-$convexity (see [17]).

Definition 1.10. Suppose $(\alpha,m) \in [0,1]^2$. A function $g : [0,\infty) \to \mathbb{R}$ is said to be $(\alpha,m)-$convex(concave), if

$$g(\tau y + m(1-\tau)z) \leq (\geq) \tau^{\alpha} g(y) + m(1-\tau^{\alpha})g(z) \tag{1.7}$$

$\forall\ y,z \in [0,\infty),\ \tau \in [0,1]$. Above function can also be written as $(m,s)-$convex(concave) function in the 1^{st} kind.

Firstly, we introduce a novel class of $(m,s)-$convex(concave) function in the 2^{nd} kind that is given below:

Definition 1.11. Let $(m,s) \in (0,1]^2$. Any function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $(m,s)-$convex(concave) in the 2^{nd} kind, if

$$g(\tau y + m(1-\tau)z) \leq (\geq) \tau^{s} g(y) + m(1-\tau)^{s} g(z) \tag{1.8}$$

$\forall\ y,z \in K,\ \tau \in [0,1]$.

Definition 1.12. [8] Let $(s,r) \in [0,1]^2$. Any function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $(s,r)-$convex(concave) in the mixed kind, if

$$g(\tau y + (1-\tau)z) \leq (\geq) \tau^{rs} g(y) + (1-\tau^{r})^{s} g(z), \tag{1.9}$$

$\forall\ z,y \in K,\ \tau \in [0,1]$.

Definition 1.13. [6] Suppose $(\alpha,\beta) \in [0,1]^2$. Any function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex(concave) in the 1^{st} kind, if

$$g(\tau y + (1-\tau)z) \leq (\geq) \tau^{\alpha} g(y) + (1-\tau^{\beta}) g(z), \tag{1.10}$$

$\forall\ z,y \in K,\ \tau \in [0,1]$.

Definition 1.14. [6] Suppose $(\alpha,\beta) \in [0,1]^2$. Any function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex(concave) in the 2^{nd} kind, if

$$g(\tau y + (1-\tau)z) \leq (\geq) \tau^{\alpha} g(y) + (1-\tau)^{\beta} g(z), \tag{1.11}$$

$\forall\ z,y \in K,\ \tau \in [0,1]$.

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Secondly, a novel class of $(m,s,r)-$convex(concave) functions is introduced in mixed kind which is given below:

Definition 1.15. Let $(m,s,r) \in [0,1]^3$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $(m,s,r)-$convex(concave) in the mixed kind, if

$$g(\tau y + m(1-\tau)z) \leq (\geq) \tau^{rs} g(y) + m(1-\tau^r)^s g(z), \qquad (1.12)$$

$\forall\, y,z \in K,\ \tau \in [0,1]$.

Thirdly, a novel class of $(m,\alpha,\beta)-$convex(concave) functions is introduced in the 1st kind which is given below:

Definition 1.16. Let $(m,\alpha,\beta) \in [0,1]^3$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m,\alpha,\beta)-$convex(concave) in the 1st kind, if

$$g(\tau y + m(1-\tau)z) \leq (\geq) \tau^{\alpha} g(y) + m(1-\tau^{\beta}) g(z), \qquad (1.13)$$

$\forall\, y,z \in K,\ \tau \in [0,1]$.

Fourthly, a novel class of $(m,\alpha,\beta)-$convex(concave) functions is introduced in the 2nd kind which is given below:

Definition 1.17. Let $(m,\alpha,\beta) \in [0,1]^3$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m,\alpha,\beta)-$convex(concave) in the 2nd kind, if

$$g(\tau y + m(1-\tau)z) \leq (\geq) \tau^{\alpha} g(y) + m(1-\tau)^{\beta} g(z), \qquad (1.14)$$

$\forall\, y,z \in K,\ \tau \in [0,1]$.

Upcoming definition is $(\alpha,\beta,\gamma,\mu)-$convex(concave) function (see [8]).

Definition 1.18. Suppose $(\alpha,\beta,\gamma,\mu) \in [0,1]^4$. Any function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(\alpha,\beta,\gamma,\mu)-$convex(concave) in the mixed kind, if

$$g(\tau y + (1-\tau)z) \leq (\geq) \tau^{\alpha\gamma} g(y) + (1-\tau^{\beta})^{\mu} g(z), \qquad (1.15)$$

$\forall\, y,z \in K,\ \tau \in [0,1]$.

Fifthly & Finally we introduce a new class of function which will be known as class of $(m,\alpha,\beta,\gamma,\mu)-$convex(concave) function in mixed kind & having whole previously said classes of functions. This terminology is applied sequentially in the article.

Definition 1.19. Suppose $(m,\alpha,\beta,\gamma,\mu) \in [0,1]^5$. Any function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m,\alpha,\beta,\gamma,\mu)-$convex(concave) in the mixed kind, if

$$g(\tau y + m(1-\tau)z) \leq (\geq) \tau^{\alpha\gamma} g(y) + m(1-\tau^{\beta})^{\mu} g(z), \qquad (1.16)$$

$\forall\, y,z \in K,\ \tau \in [0,1]$.

Remark 1.20. The following scenarios are found in Definition 1.19 as especial cases.

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(i) The $(\alpha,\beta,\gamma,\mu)-$convex(concave) function in mixed kind is obtained if choose $m = 1$ in (1.16).
(ii) The $(m,\alpha,\beta)-$convex(concave) function in 2nd kind is obtained if choose $\beta = \gamma = 1$ & $\mu = \beta$ in (1.16).
(iii) The $(m,\alpha,\beta)-$convex(concave) function in 1st kind is obtained if choose $\gamma = \mu = 1$ in (1.16).
(iv) The $(m,s,r)-$convex(concave) function in mixed kind is obtained if choose $\gamma = r$, $\alpha = \mu = s$ & $\beta = 1$ in (1.16).
(v) The $(\alpha,\beta)-$convex(concave) function in 2nd kind is obtained if choose $m = 1$, $\beta = \gamma = 1$ & $\mu = \beta$ in (1.16).
(vi) The $(\alpha,\beta)-$convex(concave) function in 1st kind is obtained if choose $m = 1$ & $\gamma = \mu = 1$ in (1.16).
(vii) The $(s,r)-$convex(concave) function in mixed kind is obtained if choose $m = 1$, $\gamma = r$, $\alpha = \mu = s$ & $\beta = 1$ in (1.16).
(viii) The $(m,s)-$convex(concave) function in 2nd kind is obtained if choose $\alpha = \mu = s$ & $\beta = \gamma = 1$ in (1.16).
(ix) The $(m,s)-$convex(concave) function in 1st kind is obtained if choose $\gamma = s$ & $\alpha = \beta = \mu = 1$ in (1.16).
(x) The $m-$convex(concave) function is obtained if choose $\alpha = \beta = \gamma = \mu = 1$ in (1.16).
(xi) The $s-$convex(concave) function in 2nd kind is obtained if choose $m = 1$, $\alpha = \mu = s$ & $\beta = \gamma = 1$ in (1.16).
(xii) The $s-$convex(concave) function in 1st kind is obtained if choose $m = 1$, $\alpha = \beta = s$ & $\gamma = \mu = 1$ in (1.16).
(xiii) The $s-$convex(concave) function in 1st kind is obtained if choose $m = 1$, $\gamma = s$ & $\alpha = \beta = \mu = 1$ in (1.16).
(xiv) The quasi$-$convex(concave) function is obtained if choose $m = 1$, $\alpha = \beta = 0$, & $\gamma = \mu = 1$ in (1.16).
(xv) The $P-$convex(concave) function is obtained if choose $m = 1$, $\alpha = \mu = 0$ & $\beta = \gamma = 1$ in (1.16).
(xvi) The ordinary convex(concave) function is obtained if choose $m = \alpha = \beta = \gamma = \mu = 1$ in (1.16).

In practically all scientific fields, inequalities play a major impact. Our primary target is on Hermite–hadamard type inequalities, despite the discipline's enormous scope.

The convexity theory is closely connected to the inequalities theory. Many well-known Inequalities in literature consequence directly from applying convex functions. The Hermite-hadamard equation is a notable equation for convexity that has been thoroughly researched in recent decades. It is discovered independently by Hadamard & Hermite and it is stated as follows: Any Function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ be convex, here $k,j \in K$ along $j<k$, If & only if

$$g\left(\frac{j+k}{2}\right) \le \frac{1}{k-j}\int_j^k g(y)dy \le \frac{g(j)+g(k)}{2}, \qquad (1.17)$$

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this is known as inequality of Hermite-Hadamard. Eq. (1.17) has become a crucial pillar in the area of probability & optimization. Additionally, numerous researchers have refined or generalised equation (1.17) for convex, s-convex, & various other varieties of functions. Regarding the history of this inequality, we must see [18].

In [5], the following consequence had derived by Agarwal & Dragomir, which includes the Hermite-Hadamard type integral equation.

Proposition 1.21. *Let* $g : K^o \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mappinG in interior* K^o *of* K*, here* $k,j \in K^o$ *along* $k > j$*. If* $|g'|$ *is convex in interval* $[j,k]$*. Then the below equation holds*

$$\left|\frac{g(j)+g(k)}{2} - \frac{1}{k-j}\int_j^k g(u)du\right| \leq \frac{(k-j)(|g'(j)|+|g'(k)|)}{8} \qquad (1.18)$$

For additional current consequences on Hermite-Hadamard type equations involving various classes of convexity, refer to [1, 3, 10, 11, 13, 15, 18, 23].

In 2011, Kavurmaci et. al. [10] established few equations of Hermite-Hadamard type for convexity & applications by utilizing Hölder inequality & Powermean inequality. In 2016, Liu et. al. [13] established few novel equations of Hermite-Hadamard type for $MT-$convexity via classical integrals & Riemann-Liouville fractional integrals, respectively. In 2023, Mehmood et. al. [14] & [16] established few novel generalised inequalities of Hermite-Hadamard type for $(s,r)-$convex functions & $(\alpha,\beta,\gamma,\mu)-$convex functions respectively & applications by Power-mean & Hölder inequality.

The primary goal of the article is to generalise few Hermite-Hadamard type inequalities to $(m,\alpha,\beta,\gamma,\mu)-$convexity in mixed kind via classical integrals by employing the H ölder & Powermean inequalities. The applications also encompass areas such as probability & numerical integration. We will capture some findings of various articles [5, 10, 14, 16] & also examine especial cases of class of $(m,\alpha,\beta,\gamma,\mu)-$convex function on different choices of $m,\alpha,\beta,\gamma,\mu$ as remarks.

2. Generalisation of Hermite-Hadamard Type Inequalities

Regarding proof of primary findings, below Lemma (see [10]) is required.

Lemma 2.1. *Let* $g : K \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping in interval* $K^o \subset \mathbb{R}$ *here* $mk, mj \in K$ *along* $mj < mk$*. If* $g' \in L[mj,mk]$*, then*

$$\frac{(mk-y)g(mk)+(y-mj)g(mj)}{k-j} - \frac{1}{k-j}\int_{mj}^{mk} g(u)du$$

$$= \frac{(y-mj)^2}{k-j}\int_o^1(\tau-1)g'(\tau y+m(1-\tau)j)d\tau + \frac{(mk-y)^2}{k-j}\int_o^1(1-\tau)g'(\tau y+m(1-\tau)k)d\tau.$$

Proof. We obtain the desired consequence by employing likewise techniques of proof of Lemma 1 of [10].

Remark 2.2. We capture Lemma 1 of [1o] if we choose $m = 1$.

The below consequences may be derived by employing Lemma 2.1

Theorem 2.3. *Let* $g : K \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping in interval* $K^{o} \subset \mathbb{R}$ ε $g' \epsilon L[mj, mk]$, *here* $mk, mj \in K$ *along* $mj<mk$. *If* $|g'|$ *is* $(m,\alpha,\beta,\gamma,\mu)-$*convex in the* $[mj,mk]$, *then*

$$\left|\frac{(mk-y)g(mk)+(y-mj)g(mj)}{k-j}-\frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$
$$\leq \frac{(y-mj)^2}{k-j}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{m|g'(j)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\frac{(mk-y)^2}{k-j}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{m|g'(k)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

for every $g \in [mj,mk]$ & $\beta > o$.

Proof. Utilizing Lemma 2.1 & taking Modulus, then

$$\left|\frac{(mk-y)g(mk)+(y-mj)g(mj)}{k-j}-\frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$
$$\leq \frac{(y-mj)^2}{k-j}\int_o^1(1-\tau)|g'(\tau y+m(1-\tau)j|d\tau+\frac{(mk-y)^2}{k-j}\int_o^1(1-\tau)|g'(\tau y+m(1-\tau)k)|d\tau$$

Since $|g'|$ is $(m,\alpha,\beta,\gamma,\mu)-$convex in mixed kind, then

$$\left|\frac{(mk-y)g(mk)+(y-mj)g(mj)}{k-j}-\frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$
$$\leq \frac{(y-mj)^2}{k-j}\int_o^1(1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|+m\left(1-\tau^{\beta}\right)^{\mu}g'(j)\right]d\tau$$
$$+\frac{(mk-y)^2}{k-j}\int_o^1(1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|+m\left(1-\tau^{\beta}\right)^{\mu}|g'(k)|\right]d\tau$$
$$=\frac{(y-mj)^2}{k-j}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{m|g'(j)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\frac{(mk-y)^2}{k-j}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{m|g'(k)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

thus completing the proof.

Note: Where B is Beta function, symbolically described as $B(l,m)=\int_o^1 \tau^{l-1}(1-\tau)^{m-1}d\tau = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$. Since $\Gamma(l)=\int_o^{\infty} e^{-u}u^{l-1}du$.

Remark 2.4. The following scenarios are found in Theorem 2.3 as especial cases.

(i) The $(m,\alpha,\beta)-$convex function in 2nd kind is obtained if choose $\beta = \gamma = 1$ & $\mu = \beta$ in Theorem 2.3.

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ISSN No: 1008-0562
Journal Of Liaoning Technical University
Natural Science Edition

(ii) The (m,α,β)−convex function in 1st kind is obtained if choose $\gamma = \mu = 1$ in Theorem 2.3.
(iii) The (m,s,r)−convex function in mixed kind is obtained if choose $\gamma = r$, $\alpha = \mu = s$ & $\beta = 1$ in Theorem 2.3.
(iv) The (α,β)−convex function in 2nd kind is obtained if choose $m = 1$, $\beta = \gamma = 1$ & $\mu = \beta$ in Theorem 2.3.
(v) The (α,β)−convex function in 1st kind is obtained if choose $m = 1$ & $\gamma = \mu = 1$ in Theorem 2.3.
(vi) The (m,s)−convex function in 2nd kind is obtained if choose $\alpha = \mu = s$ & $\beta = \gamma = 1$ in Theorem 2.3.
(vii) The (m,s)−convex function in 1st kind is obtained if choose $\gamma = s$ & $\alpha = \beta = \mu = 1$ in Theorem 2.3.
(viii) The m−convex function is obtained if choose $\alpha = \beta = \gamma = \mu = 1$ in Theorem 2.3.
(ix) The s−convex function in 2nd kind is obtained if choose $m = 1$, $\alpha = \mu = s$ & $\beta = \gamma = 1$ in Theorem 2.3.
(x) The s−convex function in 1st kind is obtained if choose $m = 1$, $\alpha = \beta = s$ & $\gamma = \mu = 1$ in Theorem 2.3.
(xi) The s−convex function in 1st kind is obtained if choose $m = 1$, $\gamma = s$ & $\alpha = \beta = \mu = 1$ in Theorem 2.3.
(xii) The quasi-convex function is obtained if choose $m = 1$, $\alpha = \beta = 0$ & $\gamma = \mu = 1$ in Theorem 2.3.
(xiii) The P−convex function is obtained if choose $m = 1$, $\alpha = \mu = 0$ & $\beta = \gamma = 1$ in Theorem 2.3.

Remark 2.5. We capture the Theorem 2.2 of [16] if we choose $m = 1$ in Theorem 2.3.

Remark 2.6. We capture the Theorem 2.2 of [14] if choose $m = 1$, $\alpha = \mu = s$, $\beta = 1$ & $\gamma = r$, where $s,r \in (0,1]$ in Theorem 2.3.

Remark 2.7. We capture the Theorem 4 of [10] if we choose $m = \alpha = \beta = \gamma = \mu = 1$ in Theorem 2.3.

Corollary 2.8. *In Theorem 2.3, choosing* $y = \frac{mj+mk}{2}$ *we get*

$$\left|m\frac{g(mj)+g(mk)}{2} - \frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$
$$\leq \frac{m}{2}\left[g(mj)\left[\frac{\left|g'\left(\frac{mj+mk}{2}\right)\right|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(j)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right]\right.$$
$$\left.+\, g(mk)\left[\frac{\left|g'\left(\frac{mj+mk}{2}\right)\right|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(k)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right]\right]$$

ISSN No: 1008-0562
Journal Of Liaoning Technical University
Natural Science Edition

Remark 2.9. Few Remarks regarding Corollary 2.8 are below as especial cases.

(i) By utilizing the convexity property of $|g'|$ in Corollary 2.8, we get established equation (1.18) (capture Theorem 2.2 of article [5]).
(ii) The $(m,\alpha,\beta)-$convex function in 2nd kind is obtained if choose $\beta = \gamma = 1$ & $\mu = \beta$ in Corollary 2.8.
(iii) The $(m,\alpha,\beta)-$convex function in 1st kind is obtained if choose $\gamma = \mu = 1$ in Corollary 2.8.
(iv) The $(m,s,r)-$convex function in mixed kind is obtained if choose $\gamma = r$, $\alpha = \mu = s$ & $\beta = 1$ in Corollary 2.8.
(v) The $(\alpha,\beta)-$convex function in 2nd kind is obtained if choose $m = 1$, $\beta = \gamma = 1$ & $\mu = \beta$ in Corollary 2.8.
(vi) The $(\alpha,\beta)-$convex function in 1st kind is obtained if choose $m = 1$ & $\gamma = \mu = 1$ in Corollary 2.8.
(vii) The $(m,s)-$convex function in 2nd kind is obtained if choose $\alpha = \mu = s$ & $\beta = \gamma = 1$ in Corollary 2.8.
(viii) The $(m,s)-$convex function in 1st kind is obtained if choose $\gamma = s$ & $\alpha = \beta = \mu = 1$ in Corollary 2.8.
(ix) The $m-$convex function is obtained if choose $\alpha = \beta = \gamma = \mu = 1$ in Corollary 2.8.
(x) The $s-$convex function in 2nd kind is obtained if choose $m = 1$, $\alpha = \mu = s$ & $\beta = \gamma = 1$ in Corollary 2.8.
(xi) The $s-$convex function in 1st kind is obtained if choose $m = 1$, $\alpha = \beta = s$ & $\gamma = \mu = 1$ in Corollary 2.8.
(xii) The $s-$convex function in 1st kind is obtained if choose $m = 1$, $\gamma = s$ & $\alpha = \beta = \mu = 1$ in Corollary 2.8.
(xiii) The quasi-convex function is obtained if choose $m = 1$, $\alpha = \beta = 0$ & $\gamma = \mu = 1$ in Corollary 2.8.
(xiv) The $P-$convex function is obtained if choose $m = 1$, $\alpha = \mu = 0$ & $\beta = \gamma = 1$ in Corollary 2.8.

Remark 2.10. The Corollary 2.6 of [16] is attained if choose $m = 1$ in Corollary 2.8.

Remark 2.11. The Corollary 2.5 of [14] is attained if choose $m = 1$, $\alpha = \mu = s$, $\beta = 1$ & $\gamma = r$, where $s,r \in (O,1]$ in Corollary 2.8.

Remark 2.12. The Corollary 2 of [10] is attained if choose $m = \alpha = \beta = \gamma = \mu = 1$ in Corollary 2.8.

Theorem 2.13. *Let $g : K \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping in interval $K^{o} \subset \mathbb{R}$ ε $g' \epsilon L[mj, mk]$, here $mk,mj \in K$ along $mj<mk$. If $|g'|^{\frac{p}{p-1}}$ is $(m,\alpha,\beta,\gamma,\mu)-$convex in the interval $[mj,mk]$ & for few fixed $1 < q$, then*

$$\left|\frac{(mk-y)g(mk)+(y-mj)g(mj)}{k-j}-\frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$

$$\leq \frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(y-mj)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)}+\frac{m|g'(j)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+(mk-y)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)}+\frac{m|g'(k)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right]$$

for every $g \in [mj,mk]$ & $\beta > 0$.

Proof. Utilizing Lemma 2.1 & $(m,\alpha,\beta,\gamma,\mu)$−convexity of $|g'|$ & then implementing the widely recognized Hölder inequality, we get

$$\left|\frac{(mk-y)g(mk)+(y-mj)g(mj)}{k-j}-\frac{1}{k-j}\int_{mj}^{mk}g(u)du\right|$$
$$\leq \frac{(y-mj)^2}{k-j}\int_o^1(1-\tau)|g'(\tau y+m(1-\tau)j|d\tau+\frac{(mk-y)^2}{k-j}\int_o^1(1-\tau)|g'(\tau y+m(1-\tau)k)|d\tau$$

$$\leq \frac{(y-mj)^2}{k-j}\int_o^1(1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|+m\left(1-\tau^\beta\right)^\mu|g'(j)|\right]d\tau$$
$$+\frac{(mk-y)^2}{k-j}\int_o^1(1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|+m\left(1-\tau^\beta\right)^\mu|g'(k)|\right]d\tau$$
$$\leq \frac{(y-mj)^2}{k-j}\left(\int_o^1(1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_o^1\left(\tau^{\alpha\gamma}|g'(y)|+m\left(1-\tau^\beta\right)^\mu|g'(j)|\right)^q d\tau\right]^{\frac{1}{p}}$$
$$+\frac{(mk-y)^2}{k-j}\left(\int_o^1(1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_o^1\left(\tau^{\alpha\gamma}|g'(y)|+m\left(1-\tau^\beta\right)^\mu|g'(k)|\right)^q d\tau\right]^{\frac{1}{p}}$$
$$\leq \frac{(y-mj)^2}{k-j}\left(\int_o^1(1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_o^1\tau^{\alpha\gamma}|g'(y)|^q d\tau+m\int_o^1\left(1-\tau^\beta\right)^\mu|g'(j)|^q d\tau\right]^{\frac{1}{p}}$$
$$+\frac{(mk-y)^2}{k-j}\left(\int_o^1(1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_o^1\tau^{\alpha\gamma}|g'(y)|^q d\tau+m\int_o^1\left(1-\tau^\beta\right)^\mu|g'(k)|^q d\tau\right]^{\frac{1}{p}}$$
$$\leq \frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(y-mj)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)}+\frac{m|g'(j)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+(mk-y)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)}+\frac{m|g'(k)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right]$$

Remark 2.14. Since we have provided remarks (i) through (xiii) for Theorem 2.3, all of the remarks employ to Theorem 2.13.

Remark 2.15. We capture the Theorem 2.1o of [16] if choose $m = 1$ in Theorem 2.13.

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Remark 2.16. We capture the Theorem 2.8 of [14] if we choose $m = 1$, $\alpha = \mu = s$, $\beta = 1$ & $\gamma = r$, where $s,r \in (0,1]$ in Theorem 2.13.
Remark 2.17. We capture the Theorem 5 of [10] if we choose $m = \alpha = \beta = \gamma = \mu = 1$ in Theorem 2.13.

Corollary 2.18. *In Theorem* 2.13*, choosing* $y = \frac{mj+mk}{2}$ *we get*

$$\left|m\frac{g(mj)+g(mk)}{2} - \frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$

$$\leq \frac{m(k-j)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{m|g'(j)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right) + \frac{\left|g'\left(\frac{mj+mk}{2}\right)\right|^q}{\alpha\gamma+1}\right)^{\frac{1}{q}} + \left(\frac{m|g'(k)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right) + \frac{\left|g'\left(\frac{mj+mk}{2}\right)\right|^q}{\alpha\gamma+1}\right)^{\frac{1}{q}}\right]$$

Remark 2.19. Since we have provided remarks (ii) through (xiv) for Corollary 2.8, all of the remarks employ to Corollary 2.18.

Remark 2.20. We capture the Corollary 2.14 of [16] if we choose $m = 1$ in Corollary 2.18.

Remark 2.21. We capture the Corollary 2.11 of [14] if we choose $m = 1$, $\alpha = \mu = s$, $\beta = 1$ & $\gamma = r$, where $s,r \in (0,1]$ in Corollary 2.18.

Remark 2.22. We recapture the Corollary 3 of [10] if we choose $m = \alpha = \beta = \gamma = \mu = 1$ in Corollary 2.18.

Theorem 2.23. *Let* $g : K \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping on* $K^o \subset \mathbb{R}$ *ε* $g' \epsilon L[mj, mk]$*, here* $mj, mk \in K$ *along* $mj<mk$. *If* $|g'|^q$ *is* $(m,\alpha,\beta,\gamma,\mu)$*−convex on* $[mj,mk]$ *& for some fixed* $1 \leq q$. *Then*

$$\left|\frac{(mk-y)g(mk)+(y-mj)g(mj)}{k-j} - \frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$

$$\leq \frac{1}{2^{1-\frac{1}{q}}(k-j)}\left[(y-mj)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(j)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+(mk-y)^2\left(\frac{|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(k)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right)^{\frac{1}{q}}\right]$$

for every $g \in [mj,mk]$, $q = \frac{p}{p-1}$ *&* $\beta > o$.

Proof. Consider $1 \leq q$ & Utilizing Lemma 2.1 & implementing the widely recognized power-mean inequality, we get

$$\left|\frac{(mk-y)g(mk)+(y-mj)g(mj)}{k-j} - \frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$

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$$\leq \frac{(y-mj)^2}{k-j}\int_o^1 (1-\tau)|g'(\tau y+m(1-\tau)j)|d\tau + \frac{(mk-y)^2}{k-j}\int_o^1 (1-\tau)|g'(\tau y+m(1-\tau)k|d\tau$$

$$\leq \frac{(y-mj)^2}{k-j}\left(\int_o^1 (1-\tau)d\tau\right)^{1-\frac{1}{q}}\left(\int_o^1 (1-\tau)|g'(\tau y+m(1-\tau)j|^q d\tau\right)^{\frac{1}{q}}$$

$$+\frac{(mk-y)^2}{k-j}\left(\int_o^1 (1-\tau)d\tau\right)^{1-\frac{1}{q}}\left(\int_o^1 (1-\tau)|g'(\tau y+m(1-\tau)k|^q d\tau\right)^{\frac{1}{q}}$$

Since $|g'|$ is $(m,\alpha,\beta,\gamma,\mu)$–convex in mixed kiNd & first taking term

$$\int_o^1 (1-\tau)|g'(\tau y+m(1-\tau)j|^q d\tau$$

$$\leq \int_o^1 (1-\tau)\left[\tau^{\alpha\gamma}|g'(y)|^q + m\left(1-\tau^{\beta}\right)^{\mu}|g'(j)|^q\right]d\tau$$

$$= \frac{|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(j)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)$$

Analogously,

$$\int_o^1 (1-\tau)|g'(\tau y+m(1-\tau)k|^q d\tau \leq \frac{|g\prime(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g\prime(k)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)$$

We obtain the desired result by combining all above inequalities.

Remark 2.24. Since we have provided remarks (i) through (xiii) for Theorem 2.3, all of the remarks employ to Theorem 2.23.
Remark 2.25. We capture the main Theorem 2.18 of [16] if we choose $m = 1$ in Theorem 2.23.
Remark 2.26. We capture the Theorem 2.14 of [14] if we choose $m = 1$, $\alpha = \mu = s$, $\beta = 1$ & $\gamma = r$, where $s,r \in (o,1]$ in Theorem 2.23.
Remark 2.27. We capture the Theorem 7 of [1o] if we choose $m = \alpha = \beta = \gamma = \mu = 1$ in Theorem 2.23.

Corollary 2.28. *In Theorem* 2.23*, choosing* $y = \frac{mj+mk}{2}$ *we get*

$$\left|m\frac{g(mj)+g(mk)}{2} - \frac{1}{k-j}\int_{mj}^{mk} g(u)du\right|$$

$$\leq m2^{\frac{1}{q}-3}(k-j)\left[\left(\frac{\left|g\prime\left(\frac{mj+mk}{2}\right)\right|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g\prime(j)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right)^{1/q} + \left(\frac{\left|g\prime\left(\frac{mj+mk}{2}\right)\right|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g\prime(k)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right)^{1/q}\right]$$

Remark 2.29. Since we have provided remarks (ii) through (xiv) for Corollary 2.8, all of the remarks employ to Corollary 2.28.
Remark 2.3o. We capture the Corollary 2.22 of [16] if we choose $m = 1$ in Corollary 2.28.

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Remark 2.31. We capture the Corollary 2.17 of [14] if we choose $m = 1$, $\alpha = \mu = s$, $\beta = 1$ & $\gamma = r$, where $s,r \in (o,1]$ in Corollary 2.28.
Remark 2.32. We capture the Corollary 4 of [1o] if we choose $m = \alpha = \beta = \gamma = \mu = 1$ in Corollary 2.28.

3. Application to Numerical Integration

3.1. **The Trapezoidal Formula.** Let $d : mj = m\theta_o < m\theta_1 < \cdots < m\theta_n = mk$ be division of interval $[mj, mk]$, here $h_i = \theta_{i+1} - \theta_i$, $(i = 0,1,2,\cdots,n-1)$ & considering the quadrature formula

$$\int_{mj}^{mk} g(u)du = Q(g,d) + R(g,d) \qquad (3.1)$$

where

$$Q(g,d) = \sum_{i=o}^{n-1}((m\theta_{i+1} - y)g(m\,\theta_{i+1}) + (y - m\theta_i)g(m\theta_i))$$

the related approximation error is represented by $R(g,d)$ for the trapezoidal variant.

Theorem 3.1. *With all the suppositions of Theorem* 2.3, *for every division* d *of the interval* $[mj,mk]$. *Then, the trapezoidal error estimate in equation (3.1) satisfies*

$$|R(g,d)| \le \sum_{i=o}^{n-1}(y - m\theta_i)^2\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(\theta_i)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\sum_{i=o}^{n-1}(m\theta_{i+1} - y)^2\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(\theta_{i+1})|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

for every $g \in [mj,mk]$ & $\beta > o$.

Proof. Utilizing Theorem 2.3 on sub-interval $[m\theta_i, m\theta_{i+1}]$ here $i = O,1,2,\ldots, n-1$ of the division, we get

$$\left|\frac{(m\theta_{i+1} - y)g(m\theta_{i+1}) + (y - m\theta_i)g(m\theta_i)}{h_i} - \frac{1}{h_i}\int_{m\theta_i}^{m\theta_{i+1}} g(u)du\right|$$
$$\le \frac{(y - m\theta_i)^2}{h_i}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(\theta_i)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\frac{(m\theta_{i+1} - y)^2}{h_i}\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m|g'(\theta_{i+1})|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$

Now we take summation over i in equation (3.1) from o to $n-1$, then

$$\left|\int_{mj}^{mk} g(u)du - Q(g,d)\right| = \left|\sum_{i=o}^{n-1}\left[\int_{m\theta_i}^{m\theta_{i+1}} g(u)du - ((m\theta_{i+1} - y)g(m\theta_{i+1}) + (y - m\theta_i)g(m\theta_i))\right]\right|$$
$$\le \sum_{i=o}^{n-1}\left|\int_{m\theta_i}^{m\theta_{i+1}} g(u)du - \left((m\theta_{i+1} - y)g(m\theta_{i+1}) + (y - m\theta_i)g(m\theta_i)\right)\right|$$

$\leq \sum_{i=o}^{n-1}(y-m\theta_i)^2\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{m|g'(\theta_i)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$

$+\sum_{i=o}^{n-1}(m\theta_{i+1}-y)^2\left[\frac{|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{m|g'(\theta_{i+1})|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$

which completes the proof.

Remark 3.2. We can also derive above similar findings for Corollary 2.8, Theorem 2.13, Corollary 2.18, Theorem 2.23 & Corollary 2.28.

Remark 3.3. The whole 2nd section's remarks are likewise applicable to Remarks 3.2 & Theorem 3.1.

4. Applications to Probability Theory

Assume *Y* is a random variable taking values in the finite $[j,k]$ along probability density function $g:[mj,mk]\to[0,1]$ & cumulative distribution function $H(y)=P(Y\leq y)=\int_{mj}^{mk}g(u)du$.

Theorem 4.1. *With all the suppositions of Theorem* 2.3*, then*

$$\left|\frac{(mk-y)H(mk)+(y-mj)H(mj)}{k-j}-\frac{mk-E(Y)}{k-j}\right|$$
$$\leq \frac{(y-mj)^2}{k-j}\left[\frac{|H'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{m|H'(j)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right]$$
$$+\frac{(mk-y)^2}{k-j}\left[\frac{|H\prime(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)}+\frac{m|H\prime(k)|}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right)-B\left(\frac{2}{\beta},\mu+1\right)\right)\right] \quad (4.1)$$

every $g\in[mj,mk]$ *&* $\beta>0$*. Here* $E(Y)$ *is expectation of Y.*

Proof. Choose g = H, we acquire (4.1), by implementing the below equation in

Theorem 2.3.

$$E(Y)=\int_{mj}^{mk}uH(u)du=mk-\int_{mj}^{mk}H(u)du.$$

∵ $H(mj)=0$ & $H(mk)=1$.

Remark 4.2. We can also derive above similar findings for Corollary 2.8, Theorem 2.13, Corollary 2.18, Theorem 2.23 & Corollary 2.28.

Remark 4.3. The whole 2nd section's remarks are likewise applicable to Remarks 4.2 & Theorem 4.1.

5. **Conclusion**

In the current paper, we have generalised few findings regarding notoriety Hermite-Hadamard type inequalities for $(m,\alpha,\beta,\gamma,\mu)-$convex functions in the mixed kind via classical integrals by implementing the widely recognized power-mean & Hölder's inequalities & applications for theory of Probability & numerical integration are also deduced. We captured various findings of several published articles [5, 10, 14, 16] & further deduced few especial cases of class of $(m,\alpha,\beta,\gamma,\mu)-$convexity on various choices of $m,\alpha,\beta,\gamma,\mu$ as remarks.

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