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HERMITE-HADAMARD LIKE INEQUALITIES FOR $s-$CONVEX FUNCTION IN THE 1ST KIND & APPLICATIONS

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Abstract. *In the present article, Hermite-Hadamard like inequalities for function whose modulus of the derivatives are $s-$convex in the 1st kind & applications for theory of Probability & numerical integration are deduced. Some consequences of several published articles would be captured as especial cases. Moreover, we deduce few especial cases of $s-$convex function.*

1. Introduction

About the features of convex functions, we code some lines from [11] 'Numerous problems in applied and pure mathematics involve convex functions. They act extremely crucial role in research of problems of non-linear and linear programming. The convex functions' theory falls under the broader topic of convexity. However, this theory significantly affects practically every area of mathematical sciences. One of the earliest areas of mathematics where the concept of convexity is necessary for graphic analysis. Calculus provides us with a useful technique, the second derivative test, to identify convexity'.

We must generalize the idea of convex functions in order to generalize Ostrowski's inequality. In this way, we may quickly identify the generalizations and specific instances of the inequality. We recollect several definitions from the literature [2] for several convex Functions.

Definition 1.1. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as convex, if

$$g(ty + (1-t)z) \leq tg(y)+(1-t)g(z), \tag{1.1}$$

$\forall\ z, y \in K,\ t \in [0,1]$.

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We remind definition of quasi-convex function from [5].

Definition 1.2. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex, if

$$g((1-t)z + ty) \leq (\geq) \max\{g(z), g(y)\} \qquad (1.2)$$

$\forall\, z, y \in K,\ t \in [0,1]$.

We present $s-$convex function in the first class (see to [10]).

Definition 1.3. Any function $g : K \subseteq [0,\infty) \to [0,\infty)$ is said to be $s-$convex in the 1st kind here $s \in (0,1]$, if

$$g(ty + (1-t)z) \leq t^s g(y) + (1-t^s) g(z), \qquad (1.3)$$

$\forall\, z, y \in K,\ t \in [0,1]$.

Remark 1.4. The following scenarios are found in Definition 1.3.

(i) The Quasi−Convex function is obtained if we choose $s = 0$ in (1.3).
(ii) The ordinary Convex function is obtained if we choose $s = 0$ in (1.3).

In practically all scientific fields, inequalities play a major impact. Our primary target is on Hermite–hadamard like inequalities, despite the discipline's enormous scope.

The convexity theory is closely connected to the inequalities theory. Many well-known Inequalities in literature consequence directly from applying convex functions. The Hermite-hadamard equation is a notable equation for convex functions that has been thoroughly researched in recent decades. It is discovered independently by Hadamard & Hermite and it is stated as follows: Any Function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ be convex, $k,j \in K$ with $k > j$, If & only if

$$g\left(\frac{j+k}{2}\right) \leq \frac{1}{k-j}\int_j^k g(y)dy \leq \frac{g(j)+g(k)}{2} \qquad (1.4)$$

this is known as Hermite-Hadamard inequality. Equation (1.4) has become a crucial pillar in the area of probability & optimization. Additionally, numerous researchers have refined or generalized equation (1.4) for convex, s-convex, quasi-convex, & various other varieties of functions.

In [4], the following consequence had derived by Agarwal & Dragomir, which includes the Hermite-Hadamard like integral equation.

Proposition 1.5. *Let* $g : K^o \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping in the Interior* K^o *of* K, *here* $k,j \in K^o$ *along* $k > j$. *If* $|g'|$ *is convex in interval* $[j,k]$. *Then the below equation holds*

$$\left|\frac{g(j)+g(k)}{2} - \frac{1}{k-j}\int_j^k g(u)du\right| \leq \frac{(k-j)(|g'(j)|+|g'(k)|)}{8} \qquad (1.5)$$

For additional recent consequences on Hermite-Hadamard like inequalities involving several classes of convex functions, refer to [1, 6, 7, 9, 14].

Kavurmaci et al. [6] established few novel inequalities 0f Hermite-Hadamard like f0r convex functions & applicati0ns by utilizing Hölder inequality and Powermean inequality.

The primary goal of the article is to generalize few Hermite-Hadamard like inequalities to $s-$c0nvex functi0ns in the 1st kind by employing the Hölder & Powermean inequalities. The applications also encompass areas such as probability & numerical integration. We will capture some findings of various articles [4, 6] and also examine especial cases of $s-$convex Function.

2. Generalization of Hermite-Hadamard Like Inequalities

Regarding proof of our primary findings, below Lemma (see [6]) is required.

Lemma 2.1. *Supp0se* $g : K \subseteq \mathbb{R} \to \mathbb{R}$ *is differentiable mapping in interval* $K^0 \subset \mathbb{R}$ *here* $k,j \in K$ *along* $k > j$, *if* $g' \in L[j,k]$, *then*

$$\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du$$

$$=\frac{(y-j)^2}{k-j}\int_0^1(1-t)g'(ty+(1-t)j)dt+\frac{(k-y)^2}{k-j}\int_0^1(1-t)g'(ty+(1-t)k)dt.$$

The below consequences may be derived by employing lemma 2.1

Theorem 2.2. *Supp0se* $g : K \subseteq \mathbb{R} \to \mathbb{R}$ *is differentiable mapping in interval* $K^o \subset \mathbb{R}\ \varepsilon\ g' \epsilon L[j,k]$, *here* $k,j \in K$ *along* $k > j$, *if* $|g'|$ *is* $s-$*c0nvex on* $[j,k]$, *then*

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{(y-j)^2}{k-j}\left[\frac{|g'(y)|}{(s+1)(s+2)}+s|g'(j)|\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right]+\frac{(k-y)^2}{k-j}\left[\frac{|g'(y)|}{(s+1)(s+2)}+s|g'(k)|\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right]$$

For every $g\epsilon[j,k]$.

Proof. Utilizing lemma 2.1 & taking Modulus, then

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{(y-j)^2}{k-j}\int_0^1(1-t)|g'((1-t)j+ty)|dt+\frac{(k-y)^2}{k-j}\int_0^1(1-t)|g'(ty+(1-t)k)|dt$$

Since $|g'|$ is $s-$convex in the 1st kind, then

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{(y-j)^2}{k-j}\int_0^1(1-t)[t^s|g'(y)|+(1-t^s)g'(j)]\,dt$$

$+\frac{(k-y)^2}{k-j}\int_0^1(1-t)\,[t^s|g'(y)|+(1-t^s)|g'(k)|]dt$
$=\frac{(y-j)^2}{k-j}\left[\frac{|g\prime(y)|}{(s+1)(s+2)}+s|g'(j)|\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right]$
$+\frac{(k-y)^2}{k-j}\left[\frac{|g\prime(y)|}{(s+1)(s+2)}+s|g'(k)|\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right]$

thus completing the proof.

Remark 2.3. The quasi−Convex function is obtained If choose s =0 in Theorem 2.2.

Remark 2.4. We capture the main Theorem 4 of [6] If choose s = 1 in Theorem 2.2.

Corollary 2.5. *In Theorem* 2.2, *choosing* $y=\frac{j+k}{2}$ *we get*

$$\left|\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g(u)du\right|\le\frac{k-j}{4}\left[\frac{2\left|g'\left(\frac{j+k}{2}\right)\right|}{(s+1)(s+2)}+s(|g'(j)|+s|g'(k)|)\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right]$$

Remark 2.6. Few Remarks regarding Corollary 2.5 are below as especial cases.

(i) By utilizing the convexity property of $|g'|$ in Corollary 2.5, we get established equation (1.5) (capture theorem 2.2 of article [4]).
(ii) The quasi−Convex function is obtained If choose s = 0 in Corollary 2.5.

Remark 2.7. We capture the Corollary 2 of article [6] If choose s = 1 in Corollary 2.5.

Theorem 2.8. *Suppose* $g: K\subseteq\mathbb{R}\to\mathbb{R}$ *is differentiable mapping in the interval* $K^o\subset\mathbb{R}$ ε $g'\epsilon L[j,k]$, *here* $k,j\in K$ *along* $k>j$. *If* $|g'|^{\frac{p}{p-1}}$ *is s−convex in the interval [j,k] & for some fixed* $1<q$, *then*

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$

$$\le\frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(y-j)^2\left(\frac{s}{s+1}|g'(j)|^q+\frac{|g'(y)|^q}{s+1}\right)^{\frac{1}{q}}+(k-y)^2\left(\frac{s}{s+1}|g'(k)|^q+\frac{|g'(y)|^q}{s+1}\right)^{\frac{1}{q}}\right]$$

for every $g\in[j,k]$.

Proof. Utilizing Lemma 2.1 & s−convexity of $|g'|$ & then implementing the widely recognized Hölder inequality, we get

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\le\frac{(y-j)^2}{k-j}\int_0^1(1-t)|g'(ty+(1-t)j|dt+\frac{(k-y)^2}{k-j}\int_0^1(1-t)|g'(ty+(1+t)k)|dt$$

$$\le\frac{(y-j)^2}{k-j}\int_0^1(1-t)\,[t^s|g'(y|+(1-t^s)|g'(j)|]dt+\frac{(k-y)^2}{k-j}\int_0^1(1-t)\,[t^s|g'(y|+(1-t^s)|g'(k)|]dt$$

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$$\leq \frac{(y-j)^2}{k-j}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1 (t^s|g'(y)|+(1-t^s)|g'(j)|)^q dt\right]^{\frac{1}{p}}$$
$$+\frac{(k-y)^2}{k-j}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1 (t^s|g'(y)|+(1-t^s)|g'(k)|)^q dt\right]^{\frac{1}{p}}$$
$$\leq \frac{(y-j)^2}{k-j}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1 t^s|g'(y)|^q dt+\int_0^1 (1-t^s)|g'(j)|^q dt\right]^{\frac{1}{p}}$$
$$+\frac{(k-y)^2}{k-j}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1 t^s|g'(y)|^q dt+\int_0^1 (1-t^s)|g'(k)|^q dt\right]^{\frac{1}{p}}$$
$$\leq \frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(y-j)^2\left(\frac{|g'(y)|^q}{s+1}+\frac{s}{s+1}|g'(j)|^q\right)^{\frac{1}{q}}+(k-y)^2\left(\frac{|g'(y)|^q}{s+1}+\frac{s}{s+1}|g'(j)|^q\right)^{\frac{1}{q}}\right]$$

Remark 2.9. The quasi−Convex function is obtained If choose *s* =0 in Theorem 2.8.

Remark 2.10. We capture the Theorem 5 0f [6] If choose *s* = 1 in Theorem 2.8.

Corollary 2.11. *In The0rem* 2.8*, choosing* $y=\frac{j+k}{2}$ *we get*

$$\left|\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g(u)du\right|\leq \frac{k-j}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{s}{s+1}|g'(j)|^q+\frac{\left|g'\left(\frac{j+k}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{s}{s+1}|g'(j)|^q+\frac{\left|g'\left(\frac{j+k}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right]$$

Remark 2.12. The quasi−Convex function is obtained If choose *s* = 0 in Corollary 2.11.

Remark 2.13. We capture the Corollary 3 of article [6] If choose *s* = 1 in Corollary 2.11.

Theorem 2.14. *Supp0se* $g: K\subseteq \mathbb{R}\to\mathbb{R}$ *is differentiable mapping on* $K^{o}\subset\mathbb{R}$ ε $g'\epsilon L[j,k]$*, here* $k,j\in K$ *along* $k>j$*. If* $|g'|^q$ *is s−c0nvex on* $[j,k]$ *& for s0me fixed* $1\leq q$*. Then*

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$
$$\leq \frac{1}{2^{1-\frac{1}{q}}(k-j)}\left[(y-j)^2\left(\frac{|g'(y)|^q}{(s+2)(s+1)}+s|g'(j)|^q\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+(k-y)^2\left(\frac{|g'(y)|^q}{(s+1)(s+2)}+s|g'(k)|^q\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right)^{\frac{1}{q}}\right]$$

for every $g\epsilon\,[j,k], q=\frac{p}{p-1}$

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Proof. Consider $1 \leq q$ & Utilizing Lemma 2.1 & implementing the widely recognized p0wer-mean inequality, we get

$$\left|\frac{(k-y)g(k)+(y-j)g(j)}{k-j}-\frac{1}{k-j}\int_j^k g(u)du\right|$$

$$\leq \frac{(k-y)^2}{k-j}\int_0^1 (1-t)|g'(ty+(1-t)j)|dt+\frac{(k-y)^2}{k-j}\int_0^1 (1-t)|g'(ty+(1-t)k|dt$$

$$\leq \frac{(y-j)^2}{k-j}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)|g'(ty+(1-t)j|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(k-y)^2}{k-j}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)|g'(ty+(1-t)k|^q dt\right)^{\frac{1}{q}}$$

Since $|g'|$ is $s-$convex in the 1st kind & first we taking term

$$\int_0^1 (1-t)|g'(ty+(1-t)j|^q dt$$

$$\leq \int_0^1 (1-t)[t^s|g'(y)|^q+(1-t^s)|g'(j)|^q]\,dt$$

$$=\frac{|g'(y)|^q}{(s+1)(s+2)}+s|g'(j)|^q\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)$$

Analogously,

$$\int_0^1 (1-t)|g'((1-t)k+ty)|^q dt \leq \frac{|g'(y)|^q}{(s+1)(s+2)}+s|g'(k)|^q\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)$$

We obtain the desired result by combining all above inequalities.

Remark 2.15. The quasi−Convex function is obtained If choose $s = 0$ in The0rem 2.14.
Remark 2.16. We capture the Theorem 7 0f article [6] If choose $s = 1$ in Theorem 2.14.

Corollary 2.17. *In Theorem* 2.14*, choosing* $y=\frac{j+k}{2}$ *we get*

$$\left|\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g(u)du\right|$$

$$\leq \frac{2^{\frac{1}{q}}}{8(k-j)}\left[\left(\frac{\left|g'\left(\frac{j+k}{2}\right)\right|^q}{(s+2)(s+1)}+s|g'(j)|^q\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right)^{1/q}\right.$$

$$\left.+\left(\frac{\left|g'\left(\frac{j+k}{2}\right)\right|^q}{(s+1)(s+2)}+s|g'(k)|^q\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right)^{1/q}\right]$$

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Remark 2.18. The quasi−Convex function is obtained If choose s =0 in Corollary 2.17.
Remark 2.19. We capture the Corollary 4 of article [6] If choose s = 1 in Corollary 2.17.

3. Application to Numerical Integration

3.1. **The Trapezoidal Formula.** Suppose $d : j = \theta_0 < \theta_1 < \cdots < \theta_n = k$ is division of interval $[j,k]$ & $h_i = \theta_{i+1} - \theta_i$, $(i = 0,1,2,\cdots,n-1)$ & consider the quadrature formula

$$\int_j^k g(u)du = Q(g,d) + R(g,d), \tag{3.1}$$

where

$$Q(g,d) = \sum_{i=0}^{n-1} ((\theta_{i+1} - y)g(\theta_{i+1}) + (y - \theta_i)g(\theta_i))$$

for the trapezoidal version & $R(g,d)$ represents the associated approximation error.

Theorem 3.1. *With all the suppositions of Theorem 2.2, for every division* d *of the interval* $[j,k]$. *Then in equation (3.1), the trapezoidal error estimate satisfies:*

$$\begin{aligned}&|R(g,d)|\\ &\leq \sum_{i=0}^{n-1}(y-\theta_i)^2\left[\frac{|g'(y)|}{(s+1)(s+2)} + s|g'(\theta_i)|\left(\frac{1}{s+1} - \frac{1}{2(s+2)}\right)\right]\\ &+\sum_{i=0}^{n-1}(\theta_{i+1}-y)^2\left[\frac{|g'(y)|}{(s+1)(s+2)} + s|g'(\theta_{i+1})|\left(\frac{1}{s+1} - \frac{1}{2(s+2)}\right)\right]\end{aligned}$$

for every $g \in [j,k]$.

Proof. Utilizing theorem 2.2 on sub-interval $[\theta_i, \theta_{i+1}]$ here i =0,1,2 ,..., $n-1$ of the division, we get

$$\begin{aligned}&\left|\frac{(\theta_{i+1}-y)g(\theta_{i+1}) + (y-\theta_i)g(\theta_i)}{h_i} - \frac{1}{h_i}\int_{\theta_i}^{\theta_{i+1}} g(u)du\right|\\ &\leq \frac{(y-\theta_i)^2}{h_i}\left[\frac{|g'(y)|}{(s+1)(s+2)} + s|g'(\theta_i)|\left(\frac{1}{s+1} - \frac{1}{2(s+2)}\right)\right] + \frac{(\theta_{i+1}-y)^2}{h_i}\left[\frac{|g'(y)|}{(s+1)(s+2)} +\right.\\ &\left. s|g'(\theta_{i+1})|\left(\frac{1}{s+1} - \frac{1}{2(s+2)}\right)\right]\end{aligned}$$

Now we take summation over i in equation (3.1) from 0 to $n-1$, then

$$\left|\int_j^k g(u)du - Q(g,d)\right| = \left|\sum_{i=0}^{n-1}\left[\int_{\theta_i}^{\theta_{i+1}} g(u)du - ((\theta_{i+1}-y)g(\theta_{i+1}) + (y-\theta_i)g(\theta_i))\right]\right|$$

$$\leq \sum_{i=0}^{n-1}\left|\int_{\theta_i}^{\theta_{i+1}} g(u)du - \left((\theta_{i+1}-y)g(\theta_{i+1}) + (y-\theta_i)g(\theta_i)\right)\right|$$

$$\leq \sum_{i=0}^{n-1}(y-\theta_i)^2\left[\frac{|g'(y)|}{(s+1)(s+2)} + s|g'(\theta_i)|\left(\frac{1}{s+1} - \frac{1}{2(s+2)}\right)\right]$$

$$+\sum_{i=0}^{n-1}(\theta_{i+1}-y)^2\left[\frac{|g\prime(y)|}{(s+1)(s+2)}+s|g'(\theta_{i+1})|\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right]$$

which completes the proof.

Remark 3.2. We can also derive above similar findings for Theorem 2.14, Corollary 2.17, Corollary 2.11, Theorem 2.8 & Corollary 2.5.

Remark 3.3. Entire Remarks of 2nd section are also hold for Remark 3.2 & Theorem 3.1.

4. Applications to Probability Theory

Suppose Y is a random variable taking values in the finite $[j,k]$ with the probability density functi0n $g : [j,k] \rightarrow [0,1]$ & the cumulative distribution function $G(y)=P(Y\leq y)=\int_j^k g(u)du$.

Theorem 4.1. *With all the assumptions of Theorem* 2.2*, then*

$$\left|\frac{(k-y)G(k)+(y-j)G(j)}{k-j}-\frac{k-E(Y)}{k-j}\right|$$

$$\leq\frac{(y-j)^2}{k-j}\left[\frac{|G\prime(y)|}{(s+1)(s+2)}+s|G'(j)|\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right]$$

$$+\frac{(k-y)^2}{k-j}\left[\frac{|G\prime(y)|}{(s+1)(s+2)}+s|G'(k)|\left(\frac{1}{s+1}-\frac{1}{2(s+2)}\right)\right] \qquad (4.1)$$

for every $g\in[j,k]$. *Here* $E(Y)$ *is the expectation of* Y.

Proof. Choose $g = G$, we obtain (4.1), by implementing the below identity in Theorem 2.2.

$$E(Y)=\int_j^k uG(u)du=k-\int_j^k G(u)du.$$

$\because G(j)=0$ & $G(k)=1$.

Remark 4.3. Entire Remarks of 2nd secti0n are also hold for Remark 4.2 & Theorem 4.1.

5. Conclusion

In the current article, the famous Hermite-Hadamard like inequalities f0r $s-$convex in the 1st kind are generalized by implementing the widely recognized power-mean & H ö lder's inequalities & applicati0ns for theory of Probability & numerical integrati0n are also deduced. We have captured various findings of several published articles [4,6] & also deduced few especial cases of class of $s-$convex function.

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