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A FAMILY OF 4-POINTS TERNARY SUBDIVISION SCHEMES BASED ON LAURANT POLYNOMIAL

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Abstract: *Computational algorithms and mathematical logic work together to create computer-aided geometric designs. CAGD uses subdivision methods for mathematical modelling and graphic design. Subdivision methods use a series of iterative procedures based on predefined rules to smooth the control polygon to a specific range of limit curves. Subdivision schemes are vital in advanced mathematical modelling, especially in computer graphics, animation, computer-aided design (CAD), and digital geometry processing. This study presented the construction of a 4-point ternary approximation sub-scheme with the help of the product of two subdivision schemes, a and b, such that* $M_m = ab$ *and analysis of its various properties. The control curve is refined, and the limit curve is achieved up to the* C^1 *continuity level. To make this research more significant, the refined curves are also shown visually in comparison to control curves.*

Keywords: *Laurant Polynomial, Computer-aided geometric designs (CAGD), 4-point ternary approximation, Computer-aided design (CAD), 4-Points Ternary Subdivision Schemes*

1. Introduction

Computing is used to help design and manufacture geometric shapes and objects, known as computer-aided geometric design (CAGD). It is a subfield of computer graphics concerned with mathematical modelling of curves, surfaces, and solids. Using mathematical algorithms and modelling methods CAGD create and connect geometric shapes in two-dimensional and three-dimensional space. Computer Graphics, animation, industrial design, and architecture are applications of CAGD. Subdivision schemes play an important role in geometric modelling and computer-aided design. This innovation helps to create smooth shapes with the help of control points. Smoother shapes are achieved in this process when new points are added at each refinement level [1-3]. In approximation schemes, the limit curve is not passed through the control vertices of the control polygon [4]. Increasing the level of refinement in subdivision schemes tends to shrink the boundary curve of the control polygon [5]. Chaikin (1974) presented a corner-cutting subdivision scheme [6]. Sabin et al. (1978) introduced a loop subdivision scheme [7]. Mustafa et al. (2013), presented tensor product subdivision [8]. Mustafa et al. (2016) presented binary subdivision techniques derived from the points of the Chaikin corner-cutting algorithm [9]. Mustafa et al. (2017) presented a cutting-edge data-driven method to choose the best subdivision schemes from flexible families using geometric and statistical model validation [10]. Mustafa et al. (2020) presented a subdivision scheme using a convolution to predict error bound in refinement procedure [11]. Mustafa et al. (2021) presented a new way to produce a parametric ternary subdivision scheme by Laurent polynomial [12]. Mustafa et al. (2023) presented a subdivision schemes with two parameters [13].

2. Construction of 4-point ternary approximating subdivision scheme M_{w}

We are taking scheme a and scheme b. After taking a product of both scheme we have a quaternary SDS. Consider smoothing operator

$$a = (1-3\beta)z^4 + \left(\frac{1}{2}-\beta\right)z^3 + (-2+4\beta)z^2 + \left(\frac{1}{2}+\beta\right)z + 1 - 3\beta \tag{1}$$

$$b = \frac{3^{m+1}\left(1+z+z^2\right)^{m+1}(1+z)^{m+2}}{2^{m+2}(3z^2)^{2m+1}} \tag{2}$$

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Now we have generate a new ternary scheme with the product of (1) and (2)

$$M_m = ab \tag{3}$$

then (3) becomes

$$M_m = \frac{1}{2^{m+2}(3z^2)^{2m+1}}[(1-3\beta)z^4 + (\tfrac{1}{2}+\beta)z^3 + (-2+4\beta)z^2 + (\tfrac{1}{2}+\beta)z + 1 - 3\beta] \tag{4}$$

2.1 A general formula of 4-point ternary subdivision scheme

Consider the laurent polynomials

$$M(z) = \frac{1}{24z^6}((1-3\beta)z^4 + (\tfrac{1}{2}+\beta)z^3 + (-2+4\beta)z^2 + (\tfrac{1}{2}+\beta)z + (1-3\beta)(1+z+z^2)^2(1+z)^3) \tag{5}$$

The mask of 4-point ternary subdivision scheme is

$$\begin{aligned} M(z) &= (\tfrac{1}{24}-\tfrac{1\beta}{8})z^5 + (\tfrac{11}{48}-\tfrac{7\beta}{12})z^4 + (\tfrac{25}{48}-\tfrac{9\beta}{8})z^3 + (\tfrac{29}{48}-\tfrac{7\beta}{8})z^2 \\ &+(\tfrac{13}{48}+\tfrac{7\beta}{12})z^1 + (\tfrac{-1}{6}+\tfrac{17\beta}{8})z^0 + (\tfrac{-1}{6}+\tfrac{17\beta}{8})z^{-1} \\ &+(\tfrac{13}{48}+\tfrac{7\beta}{12})z^{-2} + (\tfrac{29}{48}-\tfrac{7\beta}{8})z^{-3} + (\tfrac{25}{48}-\tfrac{9\beta}{8})z^{-4} \\ &+(\tfrac{11}{48}-\tfrac{7\beta}{12})z^{-5} + (\tfrac{1}{24}-\tfrac{1\beta}{8})z^{-6}. \end{aligned} \tag{6}$$

The mask of 4-point ternary subdivision scheme is

$$\begin{aligned} M_1 = \tfrac{1}{48}\{&(2-6\beta),(11-28\beta),(25-54\beta),(29-42\beta),(13+28\beta), \\ &(-8+102\beta),(-8+102\beta),(13+28\beta),(29-42\beta),(25-54\beta), \\ &(11-28\beta),(2-6\beta)\}. \end{aligned} \tag{7}$$

By using the coefficients of (6), we get the 4-points ternary subdivision scheme is

$$\begin{aligned} h^{t+1}_{3m-1} &= (\tfrac{25}{48}-\tfrac{9\beta}{8})h^t_{m-1} + (\tfrac{-1}{6}+\tfrac{17\beta}{8})h^t_m + (\tfrac{29}{48}-\tfrac{7\beta}{8})h^t_{m+1} + (\tfrac{1}{24}-\tfrac{1\beta}{8})h^t_{m+2}, \\ h^{t+1}_{3m} &= (\tfrac{11}{48}-\tfrac{7\beta}{12})h^t_{m-1} + (\tfrac{13}{48}+\tfrac{7\beta}{12})h^t_m + (\tfrac{13}{48}+\tfrac{7\beta}{12})h^t_{m+1} + (\tfrac{11}{48}-\tfrac{7\beta}{12})h^t_{m+2}, \\ h^{t+1}_{3m+1} &= (\tfrac{1}{24}-\tfrac{1\beta}{8})h^t_{m-1} + (\tfrac{29}{48}-\tfrac{7\beta}{8})h^t_m + (\tfrac{-1}{6}+\tfrac{17\beta}{8})h^t_{m+1}, \\ &+(\tfrac{25}{48}-\tfrac{9\beta}{8})h^t_{m+2}. \end{aligned} \tag{8}$$

The simplied form of (6) is

$$M(z) = \frac{1}{48}(1+z+z^2)^2\{(2-6\beta)z^0 + (7-16\beta)z^1$$

$$+(5-4\beta)z^2+(-6+26\beta)z^3+(-6+26\beta)z^4+$$
$$(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}. \quad (9)$$

3. Derivation of a family 4-point ternary approximating subdivisionscheme M_{β}.

In this area, we are going decide a family of 4-point ternary approximating subdivision conspire for $=\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and $\frac{3}{10}$.

Derivation of 4-point scheme $M_{\frac{1}{2}}$

By putting $=\frac{1}{2}$ in (7), we have mask of scheme

$$M_{\frac{1}{2}}=\frac{1}{48}\{-1,-3,-2,8,27,43,43,27,8,-2,-3,-1\}, \quad (10)$$

by putting $=\frac{1}{2}$ in (8), we get the scheme $M_{\frac{1}{2}}$ as

$$h_{3m}^{t+1}=\frac{-1}{24}h_{m-1}^{t}+\frac{43}{48}h_{m}^{t}+\frac{1}{6}h_{m+1}^{t}-\frac{1}{48}h_{m+2}^{t},$$
$$h_{3m+1}^{t+1}=\frac{-1}{16}h_{m-1}^{t}+\frac{9}{16}h_{m}^{t}+\frac{9}{16}h_{m+1}^{t}-\frac{1}{16}h_{m+2}^{t},$$
$$h_{3m+2}^{t+1}=\frac{-1}{48}h_{m-1}^{t}+\frac{1}{6}h_{m}^{t}+\frac{43}{48}h_{m+1}^{t}-\frac{1}{24}h_{m+2}^{t}. \quad (11)$$

The Laurent polynomial of proposed conspire is given as beneath

$$M_{\frac{1}{2}}(\mathrm{z})=\frac{1}{48\mathrm{z}^6}\{-1-3\mathrm{z}-2\mathrm{z}^2+8\mathrm{z}^3+27\mathrm{z}^4+43\mathrm{z}^5+43\mathrm{z}^6+27\mathrm{z}^7+8\mathrm{z}^8$$
$$-2\mathrm{z}^9-3\mathrm{z}^{10}-1\mathrm{z}^{11}\}. \quad (12)$$

Derivation of 4-point scheme $M_{\frac{1}{3}}$

By putting $=\frac{1}{3}$ in (7), we have mask of scheme

$$M_{\frac{1}{3}}=\frac{1}{144}\{0,5,21,45,67,78,78,67,45,21,5,0\}, \quad (13)$$

by putting $=\frac{1}{3}$ in (8), we get the scheme $M_{\frac{1}{3}}$ as

$$h_{3m}^{t+1}=\frac{7}{48}h_{m-1}^{t}+\frac{13}{24}h_{m}^{t}+\frac{5}{16}h_{m+1}^{t}-0h_{m+2}^{t},$$

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$$h_{3m+1}^{t+1} = \frac{5}{144}h_{m-1}^{t} + \frac{67}{144}h_{m}^{t} + \frac{67}{144}h_{m+1}^{t} + \frac{5}{144}h_{m+2}^{t},$$

$$h_{3m+2}^{t+1} = 0h_{m-1}^{t} + \frac{5}{16}h_{m}^{t} + \frac{13}{24}h_{m+1}^{t} + \frac{7}{48}h_{m+2}^{t}. \quad (14)$$

The Laurent polynomial of given scheme is

$$M_{\frac{1}{3}}(\mathrm{z}) = \frac{1}{144\mathrm{z}^6}\{0 + 5\mathrm{z} + 21\mathrm{z}^2 + 45\mathrm{z}^3 + 67\mathrm{z}^4 + 78\mathrm{z}^5 + 78\mathrm{z}^6 + 67\mathrm{z}^7$$

$$+45\mathrm{z}^8 + 21\mathrm{z}^9 + 5\mathrm{z}^{10} + 0\}. \quad (15)$$

Derivation of 4-point scheme $M_{\frac{1}{4}}$

By putting $= \frac{1}{4}$ in (7), we have mask of scheme

$$U_{\frac{1}{4}} = \frac{1}{96}\{1,8,23,37,40,35,35,40,37,23,8,1\}, \quad (16)$$

by putting $= \frac{1}{4}$ in (8), we get the scheme $M_{\frac{1}{4}}$ as

$$h_{3m}^{t+1} = \frac{23}{96}h_{m-1}^{t} + \frac{35}{96}h_{m}^{t} + \frac{37}{96}h_{m+1}^{t} + \frac{1}{96}h_{m+2}^{t},$$

$$h_{3m+1}^{t+1} = \frac{1}{12}h_{m-1}^{t} + \frac{5}{12}h_{m}^{t} + \frac{5}{12}h_{m+1}^{t} + \frac{1}{12}h_{m+2}^{t},$$

$$h_{3m+2}^{t+1} = \frac{1}{96}h_{m-1}^{t} + \frac{37}{96}h_{m}^{t} + \frac{35}{96}h_{m+1}^{t} + \frac{23}{96}h_{m+2}^{t}. \quad (17)$$

The Laurent polynomial of given scheme is

$$M_{\frac{1}{4}}(\mathrm{z}) = \frac{1}{96\mathrm{z}^6}\{1 + 8\mathrm{z} + 23\mathrm{z}^2 + 37\mathrm{z}^3 + 40\mathrm{z}^4 + 35\mathrm{z}^5 + 35\mathrm{z}^6 + 40\mathrm{z}^7 +$$

$$37\mathrm{z}^8 + 23\mathrm{z}^9 + 8\mathrm{z}^{10} + \mathrm{z}^{11}\}. \quad (18)$$

Derivation of 4-point scheme $M_{\frac{1}{5}}$

By putting $= \frac{1}{5}$ in (7), we have mask of scheme

$$M_{\frac{1}{5}} = \frac{1}{240}\{4,27,71,103,93,62,62,93,103,71,27,4\}, \quad (19)$$

by putting $= \frac{1}{5}$ in (8), we get the scheme $M_{\frac{1}{5}}$ as

$$h_{3m}^{t+1} = \frac{71}{240}h_{m-1}^{t} + \frac{31}{120}h_{m}^{t} + \frac{103}{240}h_{m+1}^{t} + \frac{1}{60}h_{m+2}^{t},$$

$$h_{3m+1}^{t+1} = \frac{9}{80}h_{m-1}^{t} + \frac{31}{60}h_{m}^{t} + \frac{31}{60}h_{m+1}^{t} + \frac{9}{80}h_{m+2}^{t},$$

$$h^{t+1}_{3m+2} = \frac{1}{60}h^t_{m-1} + \frac{103}{240}h^t_m + \frac{31}{120}h^t_{m+1} + \frac{71}{240}h^t_{m+2}. \quad (20)$$

The Laurent polynomial of proposed scheme described as

$$M_{\frac{1}{5}}(\mathrm{z}) = \frac{1}{240\mathrm{z}^6}\{4 + 27\mathrm{z} + 71\mathrm{z}^2 + 103\mathrm{z}^3 + 93\mathrm{z}^4 + 62\mathrm{z}^5 + 62\mathrm{z}^6 + 93\mathrm{z}^7$$
$$+103\mathrm{z}^8 + 71\mathrm{z}^9 + 23\mathrm{z}^{10} + 4\mathrm{z}^{11}\}. \quad (21)$$

Derivation of 4-point scheme $M_{\frac{3}{10}}$

By putting $= \frac{3}{10}$ in (7), we have mask of scheme

$$M_{\frac{3}{10}} = \frac{1}{240}\{1,13,44,82,107,113,113,107,82,44,13,1\}, \quad (22)$$

by putting $= \frac{1}{5}$ in (8), we get the scheme $M_{\frac{3}{10}}$ as

$$h^{t+1}_{3m} = \frac{11}{60}h^t_{m-1} + \frac{113}{240}h^t_m + \frac{41}{120}h^t_{m+1} + \frac{1}{240}h^t_{m+2},$$
$$h^{t+1}_{3m+1} = \frac{13}{240}h^t_{m-1} + \frac{107}{240}h^t_m + \frac{107}{240}h^t_{m+1} + \frac{13}{240}h^t_{m+2},$$
$$h^{t+1}_{3m+2} = \frac{1}{240}h^t_{m-1} + \frac{41}{120}h^t_m + \frac{113}{240}h^t_{m+1} + \frac{11}{60}h^t_{m+2}. \quad (23)$$

The Laurent polynomial of given scheme is

$$M_{\frac{3}{10}}(\mathrm{z}) = \frac{1}{240\mathrm{z}^6}\{1 + 13\mathrm{z} + 44\mathrm{z}^2 + 82\mathrm{z}^3 + 107\mathrm{z}^4 + 113\mathrm{z}^5 + 113\mathrm{z}^6 + 107\mathrm{z}^7 + 82\mathrm{z}^8 +$$
$$44\mathrm{z}^9 + 13\mathrm{z}^{10} + \mathrm{z}^{11}\}. \quad (24)$$

Table 1: Masks of our Scheme at different values of parameter.

β	Mask
$\frac{1}{2}$	$M_{\frac{1}{2}} = \frac{1}{48}\{-1,-3,-2,8,27,43,43,27,8,-2,-3,-1\}$
$\frac{1}{3}$	$M_{\frac{1}{3}} = \frac{1}{144}\{0,5,21,45,67,78,78,67,45,21,5,0\}$
$\frac{1}{4}$	$M_{\frac{1}{4}} = \frac{1}{96}\{1,8,23,37,40,35,35,40,37,23,8,1\}$

$\frac{1}{5}$	$M_{\frac{1}{5}}=\frac{1}{240}\{4,27,71,103,93,62,62,93,103,71,27,4\}$
$\frac{3}{10}$	$M_{\frac{3}{10}}=\frac{1}{240}\{1,13,44,82,107,113,113,107,82,44,13,1\}$

4. Convergence of 4-point ternary subdivision scheme of M_β.

Theorem 1: The 4-points ternary subdivision scheme (8) satisfies the necessary and sufficient conditions of convergence.

Proof. We have to prove that the necessary and sufficient condition of convergence of ternary subdivision scheme (8).

We will check $M_\beta(1)=3$ and $M_\beta(e^{p\frac{2\pi i}{3}})=0$ where p=1.

And which is substituting $z=1$ in (9) is.

$$M_\beta(1)=(1+1+1^2)^2\{\frac{1}{48}(2-6\beta)1^0+(7-16\beta)1^1+(5-4\beta)1^2+(-6+26\beta)1^3+(-6+26\beta)1^4+(5-4\beta)1^5+(7-16\beta)1^6+(2-6\beta)1^7\}.$$

$$M_\beta(1)=9\{\frac{1}{48}(2-6\beta)+(7-16\beta)+(5-4\beta)+(-6+26\beta)+(-6+26\beta)+(5-4\beta)+(7-16\beta)+(2-6\beta)\}.$$

$$M_\beta(1)=9\{\frac{1}{48}(16+0)\}.$$

$$M_\beta(1)=9\left\{\frac{1}{3}\right\}$$

After simplifying we get,

$$M_\beta(1)=3.$$

Now we check the second condition

$$M_\beta(e^{p\frac{2\pi i}{3}})=0$$

Now we solve for $M_\beta(e^{p\frac{2\pi i}{3}})=0$ where p=1.

Using De-Moivre's theorem,we get

$$z = (e^{\frac{2\pi i}{3}}) = (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -0.5 + i0.8660. \tag{25}$$

By using the (9), equation

$$M_\beta(e^{\frac{2\pi i}{3}}) = 9(1 + e^{\frac{2\pi i}{3}} + (e^{\frac{2\pi i}{3}})^2)^2\{(2-6\beta) + (7-16\beta)(e^{\frac{2\pi i}{3}})^1 + (5-4\beta)(e^{\frac{2\pi i}{3}})^2 + (-6 + 26\beta)(e^{\frac{2\pi i}{3}})^3 + (-6+26\beta)(e^{\frac{2\pi i}{3}})^4 + (5-4\beta)(e^{\frac{2\pi i}{3}})^5 + (7-16\beta)(e^{\frac{2\pi i}{3}})^6 + (2-6\beta)(e^{\frac{2\pi i}{3}})^7\}.$$

After simplify

$$M_\beta(e^{\frac{2\pi i}{3}}) = 9(1 - (0.5 + i0.8660) - (0.5 - i0.8660)^2)^2\{(2-6\beta) + (7-16\beta)(0.5 + i0.8660)^1 + (5-4\beta)(0.5+i0.8660)^2 + (-6+26\beta)(0.5+i0.8660)^3 + (-6+26\beta)(0.5+i0.8660)^4 + (5-4\beta)(0.5+i0.8660)^5 + (7-16\beta)(0.5 + i0.8660)^6 + (2-6\beta)(0.5+i0.8660)^7\}.$$

Which implies that

$$M_\beta(e^{\frac{2\pi i}{3}}) = 9(1-1-0)^2\{(2-6\beta) + (7-16\beta)(0.5+i0.8660)^1 + (5-4\beta)(0.5 + i0.8660)^2 + (-6+26\beta)(0.5+i0.8660)^3 + (-6+26\beta)(0.5+i0.8660)^4 + (5-4\beta)(0.5+i0.8660)^5 + (7-16\beta)(0.5+i0.8660)^6 + (2-6\beta)(0.5 + i0.8660)^7\}.$$

After simplifying

$$M_\beta(e^{\frac{2\pi i}{3}}) = 0.$$

Hence the 4-points ternary subdivision scheme corresponding to the simplifies form (9) satisfies the necessary condition of convergence. For sufficient conditions of the convergence if ternary subdivision scheme corresponding to simplifies form (9) then we have to prove that $\left\|(\frac{1}{3}M_\beta)\right\|_\infty < 1$ for this consider mask of the scheme (7), we have

$$M = \frac{1}{48}\{(2-6\beta), (11-28\beta), (25-54\beta), (29-42\beta), (13+28\beta), (-8+102\beta), (-8+102\beta), (13+28\beta), (29-42\beta), (25-54\beta), (11-28\beta), (2-6\beta)\}. \tag{26}$$

Now we check the sufficient condition,

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$$\left\|\left(\frac{1}{3}M_n\right)\right\|_\infty = \frac{1}{3}\, max\, \{\frac{1}{48}(|2-6\beta| + |29-42\beta| + |-8+102\beta|$$

$$+ |25-54\beta|), \frac{1}{48}(|11-28\beta| + |13+28\beta| + |13+28\beta|$$

$$+ |11-28\beta|), \frac{1}{48}(|25-54\beta| + |-8+102\beta| + |29-42\beta| + |2-6\beta|)\}.$$

After simplification, we have

$$\left\|\left(\frac{1}{3}M_\beta\right)\right\|_\infty = \; max\, \{\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\} < 1. \tag{27}$$

So the simplifies form (9) is satisfy the sufficient conditions of the convergence.

Convergence of 4-point ternary subdivision scheme at $\boldsymbol{\beta = \frac{1}{2}}$.

Theorem 2: The 4-points ternary subdivision scheme (8) satisfies the necessary and sufficient conditions of convergence.

Proof. We have to prove that the necessary and sufficient condition of convergence of ternary subdivision scheme (8).

We will check $M_{\frac{1}{2}}(1) = 3$ and $M_{\frac{1}{2}}(e^{p\,\frac{2\pi i}{3}}) = 0$ where p=1.

And which is substituting $\beta = \frac{1}{2}$ in (8) is.

$$M_{\frac{1}{2}}(z) = (1+z+z^2)^2\{\frac{1}{48}(2-6(\frac{1}{2}))z^0 + (7-16(\frac{1}{2}))z^1 + (5-4(\frac{1}{2}))z^2$$

$$+ \left(-6+26\left(\frac{1}{2}\right)\right)z^3 +$$

$$(-6+26(\tfrac{1}{2}))z^4 + (5-4(\tfrac{1}{2}))z^5 + (7-16(\tfrac{1}{2}))z^6 + (2-6(\tfrac{1}{2}))z^7\}.$$

After simplification

$$M_{\frac{1}{2}}(z) = (1+z+z^2)^2\,\frac{1}{48}(-z^7 - z^6 + 3z^5 + 7z^4 + 7z^3 + 3z^2 - z - 1) \tag{28}$$

And now substituting $z = 1$.

$$M_{\frac{1}{2}}(1) = (1+1+1^2)^2\{\frac{1}{48}(-1^7 - 1^6 + 3(1)^5 + 7(1)^4 + 7(1)^3 + 3(1)^2 - 1 - 1)\}.$$

$$M_{\frac{1}{2}}(1) = 9\{\tfrac{1}{48}(16)\}.$$

$$M_{\frac{1}{2}}(1) = 9\left\{\frac{1}{3}\right\}$$

After simplifying we get,

$$M_{\frac{1}{2}}(1) = 3.$$

Now we check the second condition

$$M_{\frac{1}{2}}(e^{p}\frac{2\pi i}{3}) = 0$$

Now we solve for $M_{\frac{1}{2}}(e^{p}\frac{2\pi i}{3}) = 0$ where $p = 1$.

Using De-Moivre's theorem,we get

$$z = (e^{\frac{2\pi i}{3}}) = (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -0.5 + i0.8660. \qquad (29)$$

Now put $z = e^{\frac{2\pi i}{3}}$ in equation (28),

$$M_{\frac{1}{2}}(e^{\frac{2\pi i}{3}}) = (1 + e^{\frac{2\pi i}{3}} + (e^{\frac{2\pi i}{3}})^2)^2 \frac{1}{48}(-(e^{\frac{2\pi i}{3}})^7 - (e^{\frac{2\pi i}{3}})^6 + 3(e^{\frac{2\pi i}{3}})^5 + 7(e^{\frac{2\pi i}{3}})^4 + 7(e^{\frac{2\pi i}{3}})^3 + 3(e^{\frac{2\pi i}{3}})^2 - (e^{\frac{2\pi i}{3}}) - 1)$$

Using (29)

$$M_{\frac{1}{2}}(e^{\frac{2\pi i}{3}}) = (1 + (-0.5 + i0.8660) + (-0.5 + i0.8660)^2)^2 \frac{1}{48}(-(-0.5 + i0.8660)^7 - (-0.5 + i0.8660)^6 + 3(-0.5 + i0.8660)^5 + 7(-0.5 + i0.8660)^4 + 7(-0.5 + i0.8660)^3 + 3(-0.5 + i0.8660)^2 - (-0.5 + i0.8660) - 1)$$

After simplify, we get

$$M_{\frac{1}{2}}(e^{\frac{2\pi i}{3}}) = 0.$$

Hence the 4-points ternary subdivision scheme corresponding to the simplifies form (9) satisfies the necessary condition of convergence. For sufficient conditions of the convergence if ternary subdivision scheme corresponding to simplifies form (9) then we have to prove that $\left\|(\frac{1}{3}M_{\frac{1}{2}})\right\|_{\infty} < 1$ for this consider mask of the scheme (7), we have

$$M_{\frac{1}{2}} = \frac{1}{48}\{-1,-3,-2,8,27,43,43,27,8,-2,-3,-1\}$$

Now we check the sufficient condition,

$\left\|\left(\frac{1}{3}M_n\right)\right\|_\infty = \frac{1}{3}\ max\ \{\frac{1}{48}(|-1| + |8| + |43| + |-2|),$

$\frac{1}{48}(|-3| + |27| + |27| + |-3|),$

$\frac{1}{48}(|-2| + |43| + |8| + |-1|)\}.$

After simplification, we have

$$\left\|\left(\frac{1}{3}M_{\frac{1}{2}}\right)\right\|_\infty = \ max\ \{\frac{1}{3},\frac{1}{3},\frac{1}{3}\} < 1. \quad (30)$$

So the simplifies form (9) is satisfy the sufficient conditions of the convergence at $\beta = \frac{1}{2}$.

Convergence of 4-point ternary subdivision scheme at $\boldsymbol{\beta = \frac{1}{3}}$.

Theorem 3: The 4-points ternary subdivision scheme (8) satisfies the necessary and sufficient conditions of convergence.

Proof. We have to prove that the necessary and sufficient condition of convergence of ternary subdivision scheme (8).

We will check $M_{\frac{1}{3}}(1) = 3$ and $M_{\frac{1}{3}}(e^{p\ \frac{2\pi i}{3}}) = 0$ where p=1.

And which is substituting $\beta = \frac{1}{3}$ in (9) is.

$$M_{\frac{1}{3}}(z) = (1 + z + z^2)^2\{\frac{1}{48}(2 - 6(\frac{1}{3}))z^0 + (7 - 16(\frac{1}{3}))z^1 + (5 - 4(\frac{1}{3}))z^2 + (-6 + 26(\frac{1}{3}))z^3 + (-6 + 26(\frac{1}{3}))z^4 + (5 - 4(\frac{1}{3}))z^5 + (7 - 16(\frac{1}{3}))z^6 + (2 - 6(\frac{1}{3}))z^7\}.$$

After simplification

$$M_{\frac{1}{3}}(z) = (1 + z + z^2)^2\frac{1}{144}(5z^6 + 11z^5 + 8z^4 + 8z^3 + 11z^2 + 5z) \quad (31)$$

And now substituting $z = 1$.

$$M_{\frac{1}{3}}(1) = (1 + 1 + 1^2)^2\frac{1}{144}(5(1)^6 + 11(1)^5 + 8(1)^4 + 8(1)^3 + 11(1)^2 + 5(1)).$$

$$M_{\frac{1}{3}}(1) = 9\{\frac{1}{144}(48)\}.$$

$$M_{\frac{1}{3}}(1) = 9\left\{\frac{1}{3}\right\}$$

After simplifying we get,

$$M_{\frac{1}{3}}(1) = 3.$$

Now we check the second condition

$$M_{\frac{1}{3}}(e^{p\,\frac{2\pi i}{3}}) = 0$$

Now we solve for $M_{\frac{1}{3}}(e^{p\,\frac{2\pi i}{3}}) = 0$ where p=1.

Using De-Moivre's theorem,we get

$$z = (e^{\frac{2\pi i}{3}}) = (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -0.5 + i0.8660. \tag{32}$$

now put $z = e^{\frac{2\pi i}{3}}$ in equation (31),

$$M_{\frac{1}{3}}(e^{\frac{2\pi i}{3}}) = (1 + e^{\frac{2\pi i}{3}} + (e^{\frac{2\pi i}{3}})^2)^2 \frac{1}{144}(5(e^{\frac{2\pi i}{3}})^6 + 11(e^{\frac{2\pi i}{3}})^5 + 8(e^{\frac{2\pi i}{3}})^4 + 8(e^{\frac{2\pi i}{3}})^3 + 11(e^{\frac{2\pi i}{3}})^2 + 5(e^{\frac{2\pi i}{3}}))$$

Using (32)

$$M_{\frac{1}{3}}(e^{\frac{2\pi i}{3}}) = (1 + (-0.5 + i0.8660) + (-0.5 + i0.8660)^2)^2 \frac{1}{48}(5(-0.5 + i0.8660)^6 + 11(-0.5 + i0.8660)^5 + 8(-0.5 + i0.8660)^4 + 8(-0.5 + i0.8660)^3 + 11(-0.5 + i0.8660)^2 + 5(-0.5 + i0.8660))$$

After simplify, we get

$$M_{\frac{1}{3}}(e^{\frac{2\pi i}{3}}) = 0.$$

Hence the 4-points ternary subdivision scheme corresponding to the simplifies form (9) satisfies the necessary condition of convergence. For sufficient conditions of the convergence if ternary subdivision scheme corresponding to simplifies form (9) then we have to prove that $\left\|(\frac{1}{3}M_{\frac{1}{3}})\right\|_{\infty} < 1$ for this consider mask of the scheme (7), we have

$$M_{\frac{1}{3}} = \frac{1}{144}\{0,5,21,45,67,78,78,67,45,21,5,0\}$$

Now we check the sufficient condition,

$$\left\|(\frac{1}{3}M_n)\right\|_\infty = \frac{1}{3}\, max\,\{\frac{1}{144}(|0|+|45|+|78|+|21|), \frac{1}{144}(|5|+|67|+|67|+|5|), \frac{1}{144}(|21| + |78|+|45|+|0|)\}.$$

After simplification, we have

$$\left\|(\frac{1}{3}M_{\frac{1}{3}})\right\|_\infty = \ max\,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\} < 1. \tag{33}$$

So the simplifies form (9) is satisfy the sufficient conditions of the convergence at $\beta = \frac{1}{3}$.

Convergence of 4-point ternary subdivision scheme at $\boldsymbol{\beta = \frac{1}{4}}$.

Theorem 4: The 4-points ternary subdivision scheme (8) satisfies the necessary and sufficient conditions of convergence.

Proof. We have to prove that the necessary and sufficient condition of convergence of ternary subdivision scheme (8).

We will check $M_{\frac{1}{4}}(1) = 3$ and $M_{\frac{1}{4}}(e^{p\,\frac{2\pi i}{3}}) = 0$ where p=1.

And which is substituting $\beta = \frac{1}{4}$ in (9) is.

$$M_{\frac{1}{4}}(z) = (1+z+z^2)^2\{\frac{1}{48}(2-6(\frac{1}{4}))z^0 + (7-16(\frac{1}{4}))z^1 + (5-4(\frac{1}{4}))z^2 + (-6+26(\frac{1}{4}))z^3 + (-6+26(\frac{1}{4}))z^4 + (5-4(\frac{1}{4}))z^5 + (7-16(\frac{1}{4}))z^6 + (2-6(\frac{1}{4}))z^7\}.$$

After simplification

$$M_{\frac{1}{4}}(z) = (1+z+z^2)^2\frac{1}{96}(z^7+6z^6+8z^5+z^4+z^3+8z^2+6z+1) \tag{34}$$

And now substituting $z = 1$.

$$M_{\frac{1}{4}}(1) = (1+1+1^2)^2\frac{1}{96}((1)^7+6(1)^6+8(1)^5+(1)^4+(1)^3+8(1)^2+6(1)+1).$$

$$M_{\frac{1}{4}}(1) = 9\{\frac{1}{96}(32)\}.$$

$$M_{\frac{1}{4}}(1) = 9\left\{\frac{1}{3}\right\}.$$

After simplifying we get,

$$M_{\frac{1}{4}}(1) = 3.$$

Now we check the second condition

$$M_{\frac{1}{4}}(e^{p}\frac{2\pi i}{3}) = 0$$

Now we solve for $M_{\frac{1}{4}}(e^{p}\frac{2\pi i}{3}) = 0$ where p=1.

Using De-Moivre's theorem,we get

$$z = (e^{\frac{2\pi i}{3}}) = (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -0.5 + i0.8660. \quad (35)$$

Now put $z = e^{\frac{2\pi i}{3}}$ in equation (34),

$$M_{\frac{1}{4}}(e^{\frac{2\pi i}{3}}) = (1 + e^{\frac{2\pi i}{3}} + (e^{\frac{2\pi i}{3}})^2)^2 \frac{1}{96}((e^{\frac{2\pi i}{3}})^7 + 6(e^{\frac{2\pi i}{3}})^6 + 8(e^{\frac{2\pi i}{3}})^5 + (e^{\frac{2\pi i}{3}})^4 + (e^{\frac{2\pi i}{3}})^3 + 8(e^{\frac{2\pi i}{3}})^2 + 6(e^{\frac{2\pi i}{3}}) + 1)$$

Using (35)

$$M_{\frac{1}{4}}(e^{\frac{2\pi i}{3}}) = (1 + (-0.5 + i0.8660) + (-0.5 + i0.8660)^2)^2 \frac{1}{96}((-0.5 + i0.8660)^7 + 6(-0.5 + i0.8660)^6 + 8(-0.5 + i0.8660)^5 + (-0.5 + i0.8660)^4 + (-0.5 + i0.8660)^3 + 8(-0.5 + i0.8660)^2 + 6(-0.5 + i0.8660) + 1)$$

After simplify, we get $M_{\frac{1}{4}}(e^{\frac{2\pi i}{3}}) = 0.$

Hence the 4-points ternary subdivision scheme corresponding to the simplifies form (9) satisfies the necessary condition of convergence. For sufficient conditions of the convergence if ternary subdivision scheme corresponding to simplifies form (9) then we have to prove that $\left\|(\frac{1}{3}M_{\frac{1}{4}})\right\|_{\infty} < 1$ for this consider mask of the scheme (7), we have

$$M_{\frac{1}{4}} = \frac{1}{96}\{1,8,23,37,40,35,35,40,37,23,8,1\}$$

Now we check the sufficient condition,

$$\left\|(\frac{1}{3}M_n)\right\|_{\infty} = \frac{1}{3}\, max\,\{\frac{1}{96}(|1| + |37| + |35| + |23|), \frac{1}{96}(|8| + |40| + |40| + |8|), \frac{1}{96}(|23| + |35| + |37| + |1|)\}.$$

After simplification, we have

$$\left\|\left(\frac{1}{3}M_{\frac{1}{4}}\right)\right\|_{\infty} = max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} < 1. \quad (36)$$

So the simplifies form (9) is satisfy the sufficient conditions of the convergence at $\beta = \frac{1}{4}$.

Convergence of 4-point ternary subdivision scheme at $\boldsymbol{\beta = \frac{1}{5}}$.

Theorem 5: The 4-points ternary subdivision scheme (8) satisfies the necessary and sufficient conditions of convergence.

Proof. We have to prove that the necessary and sufficient condition of convergence of ternary subdivision scheme (8).

We will check $M_{\frac{1}{5}}(1) = 3$ and $M_{\frac{1}{5}}(e^{p\frac{2\pi i}{3}}) = 0$ where p=1.

And which is substituting $\beta = \frac{1}{5}$ in (9) is.

$$M_{\frac{1}{5}}(z) = (1+z+z^2)^2\{\frac{1}{48}(2-6(\frac{1}{5}))z^0 + (7-16(\frac{1}{5}))z^1 + (5-4(\frac{1}{5}))z^2 + (-6+26(\frac{1}{5}))z^3 + (-6+26(\frac{1}{5}))z^4 + (5-4(\frac{1}{5}))z^5 + (7-16(\frac{1}{5}))z^6 + (2-6(\frac{1}{5}))z^7\}.$$

After simplification

$$M_{\frac{1}{5}}(z) = (1+z+z^2)^2\frac{1}{240}(4z^7+19z^6+21z^5-4z^4-4z^3+21z^2+19z+4) \quad (37)$$

And now substituting $z = 1$.

$$M_{\frac{1}{5}}(1) = (1+1+1^2)^2\frac{1}{240}(4(1)^7+19(1)^6+21(1)^5-4(1)^4-4(1)^3+21(1)^2+19(1) + 4).$$

$$M_{\frac{1}{5}}(1) = 9\{\frac{1}{240}(80)\}.$$

$$M_{\frac{1}{5}}(1) = \frac{9}{1}\{\frac{1}{3}\}.$$

After simplifying we get,

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$$M_{\frac{1}{5}}(1) = 3.$$

Now we check the second condition

$$M_{\frac{1}{5}}(e^{p}\frac{2\pi i}{3}) = 0$$

Now we solve for $M_{\frac{1}{5}}(e^{p}\frac{2\pi i}{3}) = 0$ where $p = 1$.

Using De-Moivre's theorem,we get

$$z = (e^{\frac{2\pi i}{3}}) = (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -0.5 + i0.8660. \quad (38)$$

Now put $z = e^{\frac{2\pi i}{3}}$ in equation (37),

$$M_{\frac{1}{5}}(e^{\frac{2\pi i}{3}}) = (1 + e^{\frac{2\pi i}{3}} + (e^{\frac{2\pi i}{3}})^2)^2 \frac{1}{240}(4(e^{\frac{2\pi i}{3}})^7 + 19(e^{\frac{2\pi i}{3}})^6 + 21(e^{\frac{2\pi i}{3}})^5 - 4(e^{\frac{2\pi i}{3}})^4 - 4(e^{\frac{2\pi i}{3}})^3 + 21(e^{\frac{2\pi i}{3}})^2 + 19(e^{\frac{2\pi i}{3}}) + 4)$$

Using (38)

$$M_{\frac{1}{5}}(e^{\frac{2\pi i}{3}}) = (1 + (-0.5 + i0.8660) + (-0.5 + i0.8660)^2)^2 \frac{1}{240}(4(-0.5 + i0.8660)^7 + 19(-0.5 + i0.8660)^6 + 21(-0.5 + i0.8660)^5 - 4(-0.5 + i0.8660)^4 - 4(-0.5 + i0.8660)^3 + 21(-0.5 + i0.8660)^2 + 19(-0.5 + i0.8660) + 4)$$

After simplify, we get

$$M_{\frac{1}{5}}(e^{\frac{2\pi i}{3}}) = 0.$$

Hence the 4-points ternary subdivision scheme corresponding to the simplifies form (9) satisfies the necessary condition of convergence. For sufficient conditions of the convergence if ternary subdivision scheme corresponding to simplifies form (9) then we have to prove that $\left\|(\frac{1}{3}M_{\frac{1}{5}})\right\|_{\infty} < 1$ for this consider mask of the scheme (7), we have

$$M_{\frac{1}{5}} = \frac{1}{240}\{4,27,71,103,93,62,62,93,103,71,27,4\}$$

Now we check the sufficient condition,

$$\left\|\left(\frac{1}{3}M_n\right)\right\|_\infty = \frac{1}{3}\ max\left\{\frac{1}{240}(|4|+|103|+|62|+|71|), \frac{1}{240}(|27|+|93|+|93| + |27|), \frac{1}{240}(|71|+|62|+|103|+|4|)\right\}.$$

After simplification, we have

$$\left\|\left(\frac{1}{3}M_{\frac{1}{5}}\right)\right\|_\infty = \ max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} < 1. \quad (39)$$

So the simplifies form (9) is satisfy the sufficient conditions of the convergence at $\beta = \frac{1}{5}$.

Convergence of 4-point ternary subdivision scheme at $\boldsymbol{\beta = 3/10}$.

Theorem 6: The 4-points ternary subdivision scheme (8) satisfies the necessary and sufficient conditions of convergence.

Proof. We have to prove that the necessary and sufficient condition of convergence of ternary subdivision scheme (9).

We will check $M_{\frac{3}{10}}(1) = 3$ and $M_{\frac{3}{10}}(e^{p\frac{2\pi i}{3}}) = 0$ where $p = 1$.

And which is substituting $\beta = \frac{3}{10}$ in (9) is.

$$M_{\frac{3}{10}}(z) = (1+z+z^2)^2\left\{\frac{1}{48}\left(2-6\left(\frac{3}{10}\right)\right)z^0 + \left(7-16\left(\frac{3}{10}\right)\right)z^1 + \left(5-4\left(\frac{3}{10}\right)\right)z^2 + \left(-6 + 26\left(\frac{3}{10}\right)\right)z^3 + \left(-6+26\left(\frac{3}{10}\right)\right)z^4 + \left(5-4\left(\frac{3}{10}\right)\right)z^5 + \left(7-16\left(\frac{3}{10}\right)\right)z^6 + \left(2 - 6\left(\frac{3}{10}\right)\right)z^7\right\}.$$

After simplification

$$M_{\frac{3}{10}}(z) = (1+z+z^2)^2\frac{1}{240}(z^7+11z^6+19z^5+9z^4+9z^3+19z^2+11z+1) \quad (40)$$

And now substituting $z = 1$.

$$M_{\frac{3}{10}}(1) = (1+1+1^2)^2\frac{1}{240}((1)^7+11(1)^6+19(1)^5+9(1)^4+9(1)^3+19(1)^2+11(1) + 1).$$

$$M_{\frac{3}{10}}(1) = 9\left\{\frac{1}{240}(80)\right\}.$$

$$M_{\frac{3}{10}}(1) = 9\left\{\frac{1}{3}\right\}.$$

After simplifying we get,

$$M_{\frac{3}{10}}(1) = 3.$$

Now we check the second condition

$$M_{\frac{3}{10}}(e^{p}\frac{2\pi i}{3}) = 0$$

Now we solve for $M_{\frac{3}{10}}(e^{p}\frac{2\pi i}{3}) = 0$ where p=1.

Using De-Moivre's theorem,we get

$$z = (e^{\frac{2\pi i}{3}}) = (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -0.5 + i0.8660. \quad (41)$$

Now put $z = e^{\frac{2\pi i}{3}}$ in equation (40),

$$M_{\frac{3}{10}}(e^{\frac{2\pi i}{3}}) = (1 + e^{\frac{2\pi i}{3}} + (e^{\frac{2\pi i}{3}})^2)^2 \frac{1}{240}((e^{\frac{2\pi i}{3}})^7 + 11(e^{\frac{2\pi i}{3}})^6 + 19(e^{\frac{2\pi i}{3}})^5 + 9(e^{\frac{2\pi i}{3}})^4 + 9(e^{\frac{2\pi i}{3}})^3 + 19(e^{\frac{2\pi i}{3}})^2 + 11(e^{\frac{2\pi i}{3}}) + 1)$$

Using (41)

$$M_{\frac{3}{10}}(e^{\frac{2\pi i}{3}}) = (1 + (-0.5 + i0.8660) + (-0.5 + i0.8660)^2)^2 \frac{1}{240}((-0.5 + i0.8660)^7 + 11(-0.5 + i0.8660)^6 + 19(-0.5 + i0.8660)^5 + 9(-0.5 + i0.8660)^4 + 9(-0.5 + i0.8660)^3 + 19(-0.5 + i0.8660)^2 + 11(-0.5 + i0.8660) + 1)$$

After simplify, we get

$$M_{\frac{3}{10}}(e^{\frac{2\pi i}{3}}) = 0.$$

Hence the 4-points ternary subdivision scheme corresponding to the simplifies form (9) satisfies the necessary condition of convergence. For sufficient conditions of the convergence if ternary subdivision scheme corresponding to simplifies form (9) then we have to prove that $\left\|(\frac{1}{3}M_{\frac{3}{10}})\right\|_{\infty} < 1$ for this consider mask of the scheme (7), we have

$$M_{\frac{3}{10}} = \frac{1}{240}\{1,13,44,82,107,113,113,107,82,44,13,1\}$$

Now we check the sufficient condition,

$$\left\|\left(\frac{1}{3}M_n\right)\right\|_\infty=\frac{1}{3}\ max\ \{\frac{1}{240}(|1|+|82|+|113|+|44|),\frac{1}{240}(|13|+|107|+|107|+|13|),\frac{1}{240}(|44|+|113|+|82|+|1|)\}.$$

After simplification, we have

$$\left\|\left(\frac{1}{3}M_{\frac{3}{10}}\right)\right\|_\infty=\ max\ \{\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\}<1. \tag{42}$$

So the simplifies form (9) is satisfy the sufficient conditions of the convergence at $\beta=\frac{3}{10}$.

5. Continuity analysis of a 4-point ternary subdivision scheme at M_β

Theorem 7: The 4 -point ternary subdivision scheme (8) is C^0 continuous, when $\beta\in(-0.8125,0.1875)$ and C^1 continuous, when $\beta\in(-0.64,0.02)$.

Proof. Now for C^0 continuity by using (9) can be written as

$$M_\beta^1(z)=\left(\frac{3z^2}{1+z+z^2}\right)(M_\beta)_1(z).$$

where $(M_B)_1(z)$ is defined. After putting the value of $(M_B)_1(z)$, we get

$$M_\beta^1(z)=\frac{1}{3}\left(\frac{3z^2}{1+z+z^2}\right)(1+z+z^2)^3\{(2-6\beta)z^0+(7-16\beta)z^1+(5-4\beta)z^2+(-6+26\beta)z^3+(-6+26\beta)z^4+(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}. \tag{43}$$

After simplifying

$$M_\beta^1(z)=z^2(1+z+z^2)^2\{(2-6\beta)z^0+(7-16\beta)z^1+(5-4\beta)z^2+(-6+26\beta)z^3+(-6+26\beta)z^4+(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}.$$

By collecting terms of z, we have

$$M_\beta^1(z)=\{(2-6\beta)z^0+(9-22\beta)z^1+(14-26\beta)z^2+(6+6\beta)z^3+(-7+48\beta)z^4+(-7+48\beta)z^5+(6+6\beta)z^6+(14-26\beta)z^7+(9-22\beta)z^8+(2-6\beta)z^9\}.$$

Let S_1 be the mask of scheme S_1 from (7), we have

$$S_1=\{(2-6\beta),(9-22\beta),(14-26\beta),(6+6\beta),(-7+48\beta),(-7+48\beta),(6+6\beta),(14-26\beta),(9-22\beta),(2-6\beta)\}. \tag{44}$$

For C^0 continuity of The 4 -point ternary subdivision scheme corresponding (3.9) C^0 continuity if the scheme S_1 corresponding to this S_1 scheme is contractive.For contractiveness we check $\left\|\frac{1}{3}S_1\right\|_\infty < 1$

We have to check that

$$\left\|(\tfrac{1}{3}S_n)^L\right\|_\infty = max\,\{\tfrac{1}{3^L}\textstyle\sum_j \left|\alpha^{[n,L]}_{i+(3^L_j)}\right|; i = 0,1,\ldots,(3^L)-1\}. \tag{45}$$

By $n = 1$ and $L = 1$ in (45), we have

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = max\,\{\tfrac{1}{3}\textstyle\sum_j \left|\alpha^{[1,1]}_{3j}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+1}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+2}\right|\}. \tag{46}$$

From (44), we get

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{48}(|2-6\beta|+|6+6\beta|+|6+6\beta|+|2-6\beta|),\frac{1}{48}(|9-22\beta| + |-7+48\beta|+|14-26\beta|),\frac{1}{48}(|14-26\beta|+|-7+48|+|9-22\beta|)\}.$$

After simplification, we have

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = max\,\{\tfrac{1}{48}(16),\tfrac{1}{48}(16),\tfrac{1}{48}(16)\}.$$

Which implies

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = max\,\{\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\} < 1.$$

Show that the scheme S_1 is contractive,

By $n = 1$ and $L = 1$ in (45), we have

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = max\,\{\tfrac{1}{3}\textstyle\sum_j \left|\alpha^{[1,1]}_{3j}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+1}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+2}\right|\}. \tag{47}$$

Which implies,

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{48}(|2+6\beta|+|6+6\beta|+|6+6\beta|+|2+6\beta|),\frac{1}{48}(|9+22\beta| + |7+48\beta|+|14+26\beta|),\frac{1}{48}(|14+26\beta|+|7+48|+|9+22\beta|)\}.$$

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{48}|16+24\beta|,\frac{1}{48}|30+96\beta|,\frac{1}{48}|30+96\beta|\} < 1.$$

Now we find the interval of β

$$\frac{1}{48}|16+24\beta| < 1$$

$$\Leftrightarrow -1 < \frac{16+24\beta}{48} < 1$$

$$\Leftrightarrow \frac{-64}{24} < \beta < \frac{32}{24}$$

$$\Leftrightarrow \frac{-8}{3} < \beta < \frac{4}{3}$$

OR

$$\Leftrightarrow -2.66 < \beta < 1.33$$

Now we let

$$\frac{1}{48}|30+96\beta| < 1$$

$$\Leftrightarrow -1 < \frac{30+96\beta}{48} < 1$$

$$\Leftrightarrow \frac{-78}{96} < \beta < \frac{18}{96}$$

$$\Leftrightarrow \frac{-13}{16} < \beta < \frac{3}{16}$$

OR

$$\Leftrightarrow -0.8125 < \beta < 0.1875$$

The common region of convergence is $\beta \in (-0.8125,0.1875)$.

Hence the ternary subdivision scheme presented in (8) is C^0 continuous then $\beta \in (-0.8125,0.1875)$.

Now we solve for C^1 continuity, multiplying equation (9) with $(\frac{3z^2}{1+z+z^2})$ and get,

$$M_\beta^2(z) = \left(\frac{3z^2}{1+z+z^2}\right) M_\beta^1(z).$$

where $M_\beta^1(z)$ is defined. After putting the value of $M_\beta^1(z)$, we get

$$M_\beta^2(z) = \left(\frac{3z^2}{1+z+z^2}\right) z^2(1+z+z^2)^2\{(2-6\beta)z^0 + (7-16\beta)z^1 + (5-4\beta)z^2 + (-6+$$

$26\beta)z^3+(-6+26\beta)z^4+(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}.$ (48)

After simplification we have

$$M_\beta^2(z)=3z^4(1+z+z^2)\{(2-6\beta)z^0+(7-16\beta)z^1+(5-4\beta)z^2+(-6+26\beta)z^3+(-6+26\beta)z^4+(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}.$$

By collecting terms of z, we have

$$M_\beta^2(z)=3\{(2-6\beta)z^0+(7-16\beta)z^1+(5-4\beta)z^2+(-6+26\beta)z^3+(-6+26\beta)z^4+(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}.$$

Let $\top^1$ be the mask of scheme S_2 from (7), we have

$\top^2=3\{(2-6\beta),(7-16\beta),(5-4\beta),(-6+26\beta),(-6+26\beta),(5-4\beta),(7-16\beta),(2-6\beta)\}.$ (49)

For C^1 continuity then S_2 be the subdivision scheme corresponding to the equation (45) then S_2 .

By $n=2$ and $L=1$ in (44), we have

$$\left\|\left(\tfrac{1}{3}S_1\right)\right\|_\infty^1=\ max\,\{\tfrac{1}{3}\textstyle\sum_j\left|\alpha_{3j}^{[2,1]}\right|,\ \ \tfrac{1}{3}\sum_j\left|\alpha_{3j+1}^{[2,1]}\right|,\ \ \tfrac{1}{3}\sum_j\left|\alpha_{3j+2}^{[2,1]}\right|\}. \quad (50)$$

$$\left\|\left(\frac{1}{3}S_2\right)\right\|_\infty^1=\ max\ 3\{\frac{1}{48}(|2-6\beta|+|-6+26\beta|+|7-16\beta|),$$
$$\frac{1}{48}(|7-16\beta|+|-6+26\beta|+|2-6\beta|),\frac{1}{48}(|5-4\beta|+|5-4\beta|)\}.$$

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_\infty^1=\ max\,\{\frac{1}{16}|3+4\beta|,\frac{1}{16}|3+4\beta|,\frac{1}{16}|10-8\beta|\}<1.$$

Which implies

$$\left\|\left(\tfrac{1}{3}S_1\right)\right\|_\infty^1=\ max\,\{\tfrac{3}{16},\tfrac{3}{16},\tfrac{5}{8}\}<1. \quad (51)$$

This show that scheme S_2 is contractive,

By $n=2$ and $L=1$ in (45), we have

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_\infty^1=\ max\,\{\frac{1}{3}\sum_j\left|\alpha_{3j}^{[2,1]}\right|,\ \ \frac{1}{3}\sum_j\left|\alpha_{3j+1}^{[2,1]}\right|,\ \ \frac{1}{3}\sum_j\left|\alpha_{3j+2}^{[2,1]}\right|\}.$$

After simplification we have

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$$\left\|\left(\frac{1}{3}S_2\right)\right\|_\infty^1 = max\ 3\{\frac{1}{48}(|2+6\beta|+|6+26\beta|+|7+16\beta|),\frac{1}{48}(|7+16\beta|+|6+26\beta| + |2+6\beta|),\frac{1}{48}(|5+4\beta|+|5+4\beta|)\}.$$

which implies

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_\infty^1 = max\ \{\frac{1}{16}|15+48\beta|,\frac{1}{16}|15+48\beta|,\frac{1}{16}|10+8\beta|\} < 1.$$

Now we find the interval of β

$$\frac{1}{16}|15+48\beta| < 1$$

$$\Leftrightarrow -1 < \frac{15+48\beta}{16} < 1$$

$$\Leftrightarrow -31 < 48\beta < 1$$

$$\Leftrightarrow \frac{-31}{48} < \beta < \frac{1}{48}$$

OR

$$\Leftrightarrow -0.64 < \beta < 0.02$$

Now we let

$$\frac{1}{16}|10+8\beta| < 1$$

$$\Leftrightarrow -1 < \frac{10+8\beta}{16} < 1$$

$$\Leftrightarrow -26 < 8\beta < 6$$

$$\Leftrightarrow \frac{-26}{8} < \beta < \frac{6}{8}$$

$$\Leftrightarrow \frac{-13}{4} < \beta < \frac{3}{4}$$

OR

$$\Leftrightarrow -3.25 < \beta < 0.75$$

The common region of convergence is $\beta \in (-0.64,0.02)$. Hence the scheme presented in (9) is C^1 is continuous then $\beta \in (-0.64,0.02)$.

Continuity analysis of a 4-point ternary subdivision scheme at $M_{\frac{1}{2}}$

Theorem 8: The 4-point ternary subdivision scheme (8) is C^1 continuous.

Proof. Now for C^0 continuity by using (9) can be written as

$$M^{1}_{\frac{1}{2}}(z) = \left(\frac{3z^2}{1+z+z^2}\right)(M_{\frac{1}{2}})_1(z).$$

where $(M_{\frac{1}{2}})_1(z)$ is defined. After putting the value of $(M_{\frac{1}{2}})_1(z)$, we get

$$M^{1}_{\frac{1}{2}}(z) = \frac{1}{3}\left(\frac{3z^2}{1+z+z^2}\right)(1+z+z^2)^3\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\}.$$

After simplifying

$$M^{1}_{\frac{1}{2}}(z) = z^2(1+z+z^2)^2\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\}$$

By collecting terms of z, we have

$$M^{1}_{\frac{1}{2}}(z) = \{-1-2z+z^2+9z^3+17z^4+17z^5+9z^6+z^7-2z^8-z^9\}.$$

Let S_1 be the mask of scheme S_1 from (7), we have

$$S_1 = \{-1,-2,1,9,17,17,9,1,-2,-1\}.$$

For C^0 continuity of The 4-point ternary subdivision scheme corresponding (9) C^0 continuity if the scheme S_1 corresponding to this S_1 scheme is contractive.For contractiveness we check $\left\|\frac{1}{3}S_1\right\|_\infty < 1$

We have to check that

$$\left\|(\tfrac{1}{3}S_n)^L\right\|_\infty = max\,\{\tfrac{1}{3^L}\textstyle\sum_j \left|\alpha^{[n,L]}_{i+(3^L_j)}\right|; i = 0,1,\ldots,(3^L)-1\}. \quad (52)$$

By $n=1$ and $L=1$ in (45), we have

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = max\,\{\tfrac{1}{3}\textstyle\sum_j \left|\alpha^{[1,1]}_{3j}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+1}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+2}\right|\}. \quad (53)$$

From (44), we get

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{48}(|-1|+|9|+|9|+|-1|),\frac{1}{48}(|-2|+|17|+|1|),\frac{1}{48}(|1|+|17| + |-2|)\}.$$

After simplification, we have

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{48}(16),\frac{1}{48}(16),\frac{1}{48}(16)\}.$$

Which implies

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\} < 1.$$

This shows that the scheme S_1 is contractive.

Hence the scheme presented in (9) is C^0 is continuous.

Now we solve for C^1 continuity, multiplying equation (3.9) with $(\frac{3z^2}{1+z+z^2})$ and get,

$$M^2_{\frac{1}{2}}(z) = \left(\frac{3z^2}{1+z+z^2}\right) M^1_{\frac{1}{2}}(z).$$

where $M^1_{\frac{1}{2}}(z)$ is defined. After putting the value of $M^1_{\frac{1}{2}}(z)$, we get

$$M^2_{\frac{1}{2}}(z) = \left(\frac{3z^2}{1+z+z^2}\right) z^2(1+z+z^2)^2\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\}.$$

After simplification we have

$$M^2_{\frac{1}{2}}(z) = 3z^4(1+z+z^2)\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\}.$$

By collecting terms of z, we have

$$M^2_{\frac{1}{2}}(z) = 3\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\}.$$

Let $\top^1$ be the mask of scheme S_2 from (7), we have

$$\top^2 = 3\{-1,-1,3,7,7,3,-1,-1\}. \tag{54}$$

For C^1 continuity then S_2 be the subdivision scheme corresponding to the equation (45) then S_2 .

By $n = 2$ and $L = 1$ in (44), we have

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\,\{\frac{1}{3}\sum_j \left|\alpha^{[2,1]}_{3j}\right|,\ \ \frac{1}{3}\sum_j \left|\alpha^{[2,1]}_{3j+1}\right|,\ \ \frac{1}{3}\sum_j \left|\alpha^{[2,1]}_{3j+2}\right|\}. \tag{55}$$

$$\left\|(\frac{1}{3}S_2)\right\|_{\infty}^{1} = max\;3\{\frac{1}{48}(|-1|+|7|+|-1|),\frac{1}{48}(|-1|+|7|+|-1|),\frac{1}{48}(|3|+|3|)\}.$$

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\,\{\frac{1}{16}|5|,\frac{1}{16}|5|,\frac{1}{16}|6|\} < 1.$$

Which implies

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\,\{\frac{5}{16},\frac{5}{16},\frac{3}{8}\} < 1. \tag{56}$$

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This show that scheme S_2 is contractive,

Hence the scheme presented in (9) is C^1 is continuous.

Continuity analysis of a 4-point ternary subdivision scheme at $M_{\frac{1}{3}}$

Theorem 9: The 4-point ternary subdivision scheme (8) is C^1 continuous.

Proof. Now for C^0 continuity by using (9) can be written as

$$M_{\frac{1}{3}}^1(z) = \left(\frac{3z^2}{1+z+z^2}\right)(M_{\frac{1}{3}})_1(z).$$

where $(M_{\frac{1}{3}})_1(z)$ is defined. After putting the value of $(M_{\frac{1}{3}})_1(z)$, we get

$$M_{\frac{1}{3}}^1(z) = \frac{1}{3}\left(\frac{3z^2}{1+z+z^2}\right)(1+z+z^2)^3\{\frac{5}{3}z+\frac{11}{3}z^2+\frac{8}{3}z^3+\frac{8}{3}z^4+\frac{11}{3}z^5+\frac{5}{3}z^6\}.$$

After simplifying

$$M_{\frac{1}{3}}^1(z) = z^2(1+z+z^2)^2\{\frac{5}{3}z+\frac{11}{3}z^2+\frac{8}{3}z^3+\frac{8}{3}z^4+\frac{11}{3}z^5+\frac{5}{3}z^6\}$$

By collecting terms of z, we have

$$M_{\frac{1}{3}}^1(z) = \{\frac{5}{3}z+\frac{16}{3}z^2+8z^3+9z^4+9z^5+8z^6+\frac{16}{3}z^7+\frac{5}{3}z^8\}.$$

Let S_1 be the mask of scheme S_1 from (7), we have

$$S_1 = \{\tfrac{5}{3},\ \tfrac{16}{3},\ 8,9,8,\ \tfrac{16}{3},\ \tfrac{5}{3}\}.$$

For C^0 continuity of The 4-point ternary subdivision scheme corresponding (9) C^0 continuity if the scheme S_1 corresponding to this S_1 scheme is contractive.For contractiveness we check $\left\|\frac{1}{3}S_1\right\|_\infty < 1$

We have to check that

$$\left\|(\tfrac{1}{3}S_n)^L\right\|_\infty = max\,\{\tfrac{1}{3^L}\textstyle\sum_j \left|\alpha_{i+(3_j^L)}^{[n,L]}\right|; i = 0,1,\ldots,(3^L)-1\}. \tag{57}$$

By $n=1$ and $L=1$ in (45), we have

$$\left\|(\tfrac{1}{3}S_1)\right\|_\infty^1 = max\,\{\tfrac{1}{3}\textstyle\sum_j \left|\alpha_{3j}^{[1,1]}\right|,\ \ \tfrac{1}{3}\sum_j \left|\alpha_{3j+1}^{[1,1]}\right|,\ \ \tfrac{1}{3}\sum_j \left|\alpha_{3j+2}^{[1,1]}\right|\}. \tag{58}$$

From (44), we get

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\{\frac{1}{48}(\left|\frac{5}{3}\right| + |9| + \left|\frac{5}{3}\right|), \frac{1}{48}(\left|\frac{16}{3}\right| + |8|), \frac{1}{48}(|8| + \left|\frac{16}{3}\right|)\}.$$

After simplification, we have

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\{\frac{1}{48}(\frac{37}{3}), \frac{1}{48}(\frac{40}{3}), \frac{1}{48}(\frac{40}{3})\}.$$

Which implies

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\{\frac{37}{144}, \frac{5}{18}, \frac{5}{18}\} < 1.$$

This shows that the scheme S_1 is contractive.

Hence the scheme presented in (9) is C^0 is continuous.

Now we solve for C^1 continuity, multiplying equation (3.9) with $(\frac{3z^2}{1+z+z^2})$ and get,

$$M_{\frac{1}{3}}^{2}(z) = \left(\frac{3z^2}{1+z+z^2}\right) M_{\frac{1}{3}}^{1}(z).$$

where $M_{\frac{1}{3}}^{1}(z)$ is defined. After putting the value of $M_{\frac{1}{3}}^{1}(z)$, we get

$$M_{\frac{1}{3}}^{2}(z) = \left(\frac{3z^2}{1+z+z^2}\right) z^2(1+z+z^2)^2\{\frac{5}{3}z + \frac{11}{3}z^2 + \frac{8}{3}z^3 + \frac{8}{3}z^4 + \frac{11}{3}z^5 + \frac{5}{3}z^6\}.$$

After simplification we have

$$M_{\frac{1}{3}}^{2}(z) = 3z^4(1+z+z^2)\{\frac{5}{3}z + \frac{11}{3}z^2 + \frac{8}{3}z^3 + \frac{8}{3}z^4 + \frac{11}{3}z^5 + \frac{5}{3}z^6\}.$$

By collecting terms of z, we have

$$M_{\frac{1}{3}}^{2}(z) = 3\{\frac{5}{3}z + \frac{11}{3}z^2 + \frac{8}{3}z^3 + \frac{8}{3}z^4 + \frac{11}{3}z^5 + \frac{5}{3}z^6\}.$$

Let T^1 be the mask of scheme S_2 from (7), we have

$$\mathsf{T}^2 = 3\{\frac{5}{3}, \frac{11}{3}, \frac{8}{3}, \frac{8}{3}, \frac{11}{3}, \frac{5}{3}\}. \quad (59)$$

For C^1 continuity then S_2 be the subdivision scheme corresponding to S_2 .

By $n = 2$ and $L = 1$ in (45), we have

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\{\frac{1}{3}\sum_j \left|\alpha_{3j}^{[2,1]}\right|, \frac{1}{3}\sum_j \left|\alpha_{3j+1}^{[2,1]}\right|, \frac{1}{3}\sum_j \left|\alpha_{3j+2}^{[2,1]}\right|\}. \quad (60)$$

$$\left\|\left(\frac{1}{3}S_2\right)\right\|_{\infty}^{1} = max\,3\{\frac{1}{48}(\left|\frac{5}{3}\right| + \left|\frac{8}{3}\right|), \frac{1}{48}(\left|\frac{11}{3}\right| + \left|\frac{11}{3}\right|), \frac{1}{48}(\left|\frac{8}{3}\right| + \left|\frac{5}{3}\right|)\}.$$

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_{\infty}^{1} = max\,\{\frac{1}{16}\left|\frac{13}{3}\right|, \frac{1}{16}\left|\frac{22}{3}\right|, \frac{1}{16}\left|\frac{13}{3}\right|\} < 1.$$

Which implies

$$\left\|\left(\tfrac{1}{3}S_1\right)\right\|_{\infty}^{1} = max\,\{\tfrac{13}{48}, \tfrac{11}{24}, \tfrac{13}{48}\} < 1. \quad (61)$$

This show that scheme S_2 is contractive,

Hence the scheme presented in (9) is C^1 is continuous.

Continuity analysis of a 4-point ternary subdivision scheme at $M_{\frac{1}{4}}$

Theorem 10: The 4-point ternary subdivision scheme (8) is C^1 continuous.

Proof. Now for C^0 continuity by using (9) can be written as

$$M_{\frac{1}{4}}^{1}(z) = \left(\frac{3z^2}{1+z+z^2}\right)(M_{\frac{1}{4}})_1(z).$$

where $(M_{\frac{1}{4}})_1(z)$ is defined. After putting the value of $(M_{\frac{1}{4}})_1(z)$, we get

$$M_{\frac{1}{4}}^{1}(z) = \frac{1}{3}\left(\frac{3z^2}{1+z+z^2}\right)(1+z+z^2)^3\{\frac{1}{2} + 3z + 4z^2 + \frac{1}{2}z^3 + \frac{1}{2}z^4 + 4z^5 + 3z^6 + \frac{1}{2}z^7\}.$$

After simplifying

$$M_{\frac{1}{2}}^{1}(z) = z^2(1+z+z^2)^2\{\frac{1}{2} + 3z + 4z^2 + \frac{1}{2}z^3 + \frac{1}{2}z^4 + 4z^5 + 3z^6 + \frac{1}{2}z^7\}$$

By collecting terms of z, we have

$$M_{\frac{1}{4}}^{1}(z) = \{\frac{1}{2} + \frac{7}{2}z + \frac{15}{2}z^2 + \frac{15}{2}z^3 + 5z^4 + 5z^5 + \frac{15}{2}z^6 + \frac{15}{2}z^7 + \frac{7}{2}z^8 + \frac{1}{2}z^9\}.$$

Let S_1 be the mask of scheme S_1 from (7), we have

$$S_1 = \{\frac{1}{2}, \frac{7}{2}, \frac{15}{2}, \frac{15}{2}, 5, 5, \frac{15}{2}, \frac{15}{2}, \frac{7}{2}, \frac{1}{2}\}.$$

For C^0 continuity of The 4-point ternary subdivision scheme corresponding (9) C^0 continuity if the scheme S_1 corresponding to this S_1 scheme is contractive.For contractiveness we check $\left\|\frac{1}{3}S_1\right\|_{\infty} < 1$

We have to check that

$$\left\|(\tfrac{1}{3}S_n)^L\right\|_\infty = max\,\{\tfrac{1}{3^L}\textstyle\sum_j \left|\alpha^{[n,L]}_{i+(3^L_j)}\right|; i=0,1,\ldots,(3^L)-1\}. \quad (62)$$

By $n=1$ and $L=1$ in (45), we have

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = max\,\{\tfrac{1}{3}\textstyle\sum_j \left|\alpha^{[1,1]}_{3j}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+1}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+2}\right|\}. \quad (63)$$

From (44), we get

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{48}(\left|\frac{1}{2}\right|+\left|\frac{15}{2}\right|+\left|\frac{15}{2}\right|+\left|\frac{1}{2}\right|),\frac{1}{48}(\left|\frac{7}{2}\right|+|5|+\left|\frac{15}{2}\right|),\frac{1}{48}(\left|\frac{15}{2}\right|+|5|+\left|\frac{7}{2}\right|)\}.$$

After simplification, we have

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{48}(\frac{32}{2}),\frac{1}{48}(\frac{32}{2}),\frac{1}{48}(\frac{32}{2})\}.$$

Which implies

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\} < 1.$$

This shows that the scheme S_1 is contractive.

Hence the scheme presented in (9) is C^0 is continuous.

Now we solve for C^1 continuity, multiplying equation (3.9) with $(\frac{3z^2}{1+z+z^2})$ and get,

$$M^2_{\frac{1}{4}}(z) = \left(\frac{3z^2}{1+z+z^2}\right)M^1_{\frac{1}{4}}(z).$$

where $M^1_{\frac{1}{4}}(z)$ is defined. After putting the value of $M^1_{\frac{1}{4}}(z)$, we get

$$M^2_{\frac{1}{4}}(z) = \left(\frac{3z^2}{1+z+z^2}\right)z^2(1+z+z^2)^2\{\frac{1}{2}+3z+4z^2+\frac{1}{2}z^3+\frac{1}{2}z^4+4z^5+3z^6+\frac{1}{2}z^7\}.$$

After simplification we have

$$M^2_{\frac{1}{4}}(z) = 3z^4(1+z+z^2)\{\frac{1}{2}+3z+4z^2+\frac{1}{2}z^3+\frac{1}{2}z^4+4z^5+3z^6+\frac{1}{2}z^7\}.$$

By collecting terms of z, we have

$$M^2_{\frac{1}{4}}(z) = 3\{\frac{1}{2}+3z+4z^2+\frac{1}{2}z^3+\frac{1}{2}z^4+4z^5+3z^6+\frac{1}{2}z^7\}.$$

Let $\top^1$ be the mask of scheme S_2 from (7), we have

$$\mathrm{T}^2 = 3\{\tfrac{1}{2}, 3,4, \tfrac{1}{2}, \tfrac{1}{2}, 4,3, \tfrac{1}{2}\}. \tag{64}$$

For C^1 continuity then S_2 be the subdivision scheme corresponding to S_2 .

By $n = 2$ and $L = 1$ in (45), we have

$$\left\|(\tfrac{1}{3}S_1)\right\|_{\infty}^{1} = max\,\{\tfrac{1}{3}\textstyle\sum_j \left|\alpha_{3j}^{[2,1]}\right|,\ \ \tfrac{1}{3}\sum_j \left|\alpha_{3j+1}^{[2,1]}\right|,\ \ \tfrac{1}{3}\sum_j \left|\alpha_{3j+2}^{[2,1]}\right|\}. \tag{65}$$

$$\left\|(\frac{1}{3}S_2)\right\|_{\infty}^{1} = max\ 3\{\frac{1}{48}(\left|\frac{1}{2}\right| + \left|\frac{1}{2}\right| + |3|), \frac{1}{48}(|3| + \left|\frac{1}{2}\right| + \left|\frac{1}{2}\right|), \frac{1}{48}(|4| + |4|)\}.$$

$$\left\|(\frac{1}{3}S_1)\right\|_{\infty}^{1} = max\,\{\frac{1}{16}|4|, \frac{1}{16}|4|, \frac{1}{16}|8|\} < 1.$$

Which implies

$$\left\|(\tfrac{1}{3}S_1)\right\|_{\infty}^{1} = max\,\{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\} < 1. \tag{66}$$

This show that scheme S_2 is contractive,

Hence the scheme presented in (9) is C^1 is continuous.

Continuity analysis of a 4-point ternary subdivision scheme at $M_{\frac{1}{5}}$

Theorem 11: The 4-point ternary subdivision scheme (8) is C^1 continuous.

Proof. Now for C^0 continuity by using (9) can be written as

$$M_{\frac{1}{5}}^{1}(z) = \left(\frac{3z^2}{1+z+z^2}\right)(M_{\frac{1}{5}})_1(z).$$

where $(M_{\frac{1}{5}})_1(z)$ is defined. After putting the value of $(M_{\frac{1}{5}})_1(z)$, we get

$$M_{\frac{1}{5}}^{1}(z) = \frac{1}{3}\left(\frac{3z^2}{1+z+z^2}\right)(1+z+z^2)^3\{\frac{4}{5} + \frac{19}{5}z + \frac{21}{5}z^2 - \frac{4}{5}z^3 - \frac{4}{5}z^4 + \frac{21}{5}z^5 + \frac{19}{5}z^6 + \frac{4}{5}z^7\}.$$

After simplifying

$$M_{\frac{1}{5}}^{1}(z) = z^2(1+z+z^2)^2\{\frac{4}{5} + \frac{19}{5}z + \frac{21}{5}z^2 - \frac{4}{5}z^3 - \frac{4}{5}z^4 + \frac{21}{5}z^5 + \frac{19}{5}z^6 + \frac{4}{5}z^7\}$$

By collecting terms of z, we have

$$M_{\frac{1}{5}}^{1}(z) = \{\frac{4}{5} + \frac{23}{5}z + \frac{44}{5}z^2 + \frac{36}{5}z^3 + \frac{13}{5}z^4 + \frac{13}{5}z^5 + \frac{36}{5}z^6 + \frac{44}{5}z^7 + \frac{23}{5}z^8 + \frac{4}{5}z^9\}.$$

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Let S_1 be the mask of scheme S_1 from (7), we have

$$S_1 = \{\frac{4}{5}, \frac{23}{5}, \frac{44}{5}, \frac{36}{5}, \frac{13}{5}, \frac{13}{5}, \frac{36}{5}, \frac{44}{5}, \frac{23}{5}, \frac{4}{5}\}.$$

For C^0 continuity of The 4-point ternary subdivision scheme corresponding (9) C^0 continuity if the scheme S_1 corresponding to this S_1 scheme is contractive.For contractiveness we check $\left\|\frac{1}{3}S_1\right\|_\infty < 1$

We have to check that

$$\left\|(\frac{1}{3}S_n)^L\right\|_\infty = max\, \{\frac{1}{3^L}\sum_j \left|\alpha^{[n,L]}_{i+(3^L_j)}\right|; i = 0,1,\ldots,(3^L) - 1\}. \quad (67)$$

By $n = 1$ and $L = 1$ in (45), we have

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\, \{\frac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j}\right|, \ \ \frac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+1}\right|, \ \ \frac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+2}\right|\}. \quad (68)$$

From (44), we get

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\, \{\frac{1}{48}(\left|\frac{4}{5}\right| + \left|\frac{36}{5}\right| + \left|\frac{36}{5}\right| + \left|\frac{4}{5}\right|), \frac{1}{48}(\left|\frac{23}{5}\right| + \left|\frac{13}{5}\right| + \left|\frac{44}{5}\right|), \frac{1}{48}(\left|\frac{44}{5}\right| + \left|\frac{13}{5}\right| + \left|\frac{23}{5}\right|)\}.$$

After simplification, we have

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\, \{\frac{1}{48}(16), \frac{1}{48}(16), \frac{1}{48}(16)\}.$$

Which implies

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} < 1.$$

This shows that the scheme S_1 is contractive.

Hence the scheme presented in (9) is C^0 is continuous.

Now we solve for C^1 continuity, multiplying equation (9) with $(\frac{3z^2}{1+z+z^2})$ and get,

$$M^2_{\frac{1}{5}}(z) = \left(\frac{3z^2}{1+z+z^2}\right) M^1_{\frac{1}{5}}(z).$$

where $M^1_{\frac{1}{5}}(z)$ is defined. After putting the value of $M^1_{\frac{1}{5}}(z)$, we get

$$M^2_{\frac{1}{5}}(z) = \left(\frac{3z^2}{1+z+z^2}\right) z^2(1+z+z^2)^2\{\frac{4}{5}+\frac{19}{5}z+\frac{21}{5}z^2-\frac{4}{5}z^3-\frac{4}{5}z^4+\frac{21}{5}z^5+\frac{19}{5}z^6 + \frac{4}{5}z^7\}.$$

After simplification we have

$$M^2_{\frac{1}{2}}(z) = 3z^4(1+z+z^2)\{\frac{4}{5}+\frac{19}{5}z+\frac{21}{5}z^2-\frac{4}{5}z^3-\frac{4}{5}z^4+\frac{21}{5}z^5+\frac{19}{5}z^6+\frac{4}{5}z^7\}.$$

By collecting terms of z, we have

$$M^2_{\frac{1}{5}}(z) = 3\{\frac{4}{5}+\frac{19}{5}z+\frac{21}{5}z^2-\frac{4}{5}z^3-\frac{4}{5}z^4+\frac{21}{5}z^5+\frac{19}{5}z^6+\frac{4}{5}z^7\}.$$

Let T^1 be the mask of scheme S_2 from (3.7), we have

$$\mathrm{T}^2 = 3\{\tfrac{4}{5},\tfrac{19}{5},\tfrac{21}{5},\tfrac{-4}{5},\tfrac{-4}{5},\tfrac{21}{5},\tfrac{19}{5},\tfrac{4}{5}\}. \quad (69)$$

For C^1 continuity then S_2 be the subdivision scheme corresponding to S_2 .

By $n = 2$ and $L = 1$ in (45), we have

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = max\,\{\tfrac{1}{3}\textstyle\sum_j \left|\alpha^{[2,1]}_{3j}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[2,1]}_{3j+1}\right|,\ \tfrac{1}{3}\sum_j \left|\alpha^{[2,1]}_{3j+2}\right|\}. \quad (70)$$

$$\left\|(\frac{1}{3}S_2)\right\|^1_\infty = max\ 3\{\frac{1}{48}(\left|\frac{4}{5}\right|+\left|\frac{-4}{5}\right|+\left|\frac{19}{5}\right|),\frac{1}{48}(\left|\frac{19}{5}\right|+\left|\frac{-4}{5}\right|+\left|\frac{4}{5}\right|),\frac{1}{48}(\left|\frac{21}{5}\right|+\left|\frac{21}{5}\right|)\}.$$

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = max\,\{\frac{1}{16}\left|\frac{19}{5}\right|,\frac{1}{16}\left|\frac{19}{5}\right|,\frac{1}{16}\left|\frac{42}{5}\right|\} < 1.$$

Which implies

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = max\,\{\tfrac{19}{80},\tfrac{19}{80},\tfrac{21}{40}\} < 1. \quad (71)$$

This show that scheme S_2 is contractive,

Hence the scheme presented in (9) is C^1 is continuous.

Continuity analysis of a 4-point ternary subdivision scheme at $M_{\frac{3}{10}}$

Theorem 12: The 4-point ternary subdivision scheme (??) is C^1 continuous.

Proof. Now for C^0 continuity by using (??) can be written as

$$M^1_{\frac{3}{10}}(z) = \left(\frac{3z^2}{1+z+z^2}\right)(M_{\frac{3}{10}})_1(z).$$

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where $(M_{\frac{3}{10}})_1(z)$ is defined. After putting the value of $(M_{\frac{3}{10}})_1(z)$, we get

$$M^1_{\frac{3}{10}}(z) = \frac{1}{3}\left(\frac{3z^2}{1+z+z^2}\right)(1+z+z^2)^3\{\frac{1}{5}+\frac{11}{5}z+\frac{19}{5}z^2+\frac{9}{5}z^3+\frac{9}{5}z^4+\frac{19}{5}z^5+\frac{11}{5}z^6 + \frac{1}{5}z^7\}.$$

After simplifying

$$M^1_{\frac{3}{10}}(z) = z^2(1+z+z^2)^2\{\frac{1}{5}+\frac{11}{5}z+\frac{19}{5}z^2+\frac{9}{5}z^3+\frac{9}{5}z^4+\frac{19}{5}z^5+\frac{11}{5}z^6+\frac{1}{5}z^7\}$$

By collecting terms of z, we have

$$M^1_{\frac{3}{10}}(z) = \{\frac{1}{5}+\frac{12}{5}z+\frac{31}{5}z^2+\frac{39}{5}z^3+\frac{37}{5}z^4+\frac{37}{5}z^5+\frac{39}{5}z^6+\frac{31}{5}z^7+\frac{12}{5}z^8+\frac{1}{5}z^9\}.$$

Let S_1 be the mask of scheme S_1 from (??), we have

$$S_1 = \{\frac{1}{5},\frac{12}{5},\frac{31}{5},\frac{39}{5},\frac{37}{5},\frac{37}{5},\frac{39}{5},\frac{31}{5},\frac{12}{5},\frac{1}{5}\}.$$

For C^0 continuity of The 4-point ternary subdivision scheme corresponding (9) C^0 continuity if the scheme S_1 corresponding to this S_1 scheme is contractive.For contractiveness we check $\left\|\frac{1}{3}S_1\right\|_\infty < 1$

We have to check that

$$\left\|(\tfrac{1}{3}S_n)^L\right\|_\infty = \ max\,\{\tfrac{1}{3^L}\textstyle\sum_j \left|\alpha^{[n,L]}_{i+(3^L_j)}\right|; i = 0,1,\dots,(3^L)-1\}. \tag{72}$$

By $n = 1$ and $L = 1$ in (45), we have

$$\left\|(\tfrac{1}{3}S_1)\right\|^1_\infty = \ max\,\{\tfrac{1}{3}\textstyle\sum_j \left|\alpha^{[1,1]}_{3j}\right|,\ \ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+1}\right|,\ \ \tfrac{1}{3}\sum_j \left|\alpha^{[1,1]}_{3j+2}\right|\}. \tag{73}$$

From (44), we get

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = \ max\,\{\frac{1}{48}(\left|\frac{1}{5}\right|+\left|\frac{39}{5}\right|+\left|\frac{39}{5}\right|+\left|\frac{1}{5}\right|),\frac{1}{48}(\left|\frac{12}{5}\right|+\left|\frac{37}{5}\right|+\left|\frac{31}{5}\right|),\frac{1}{48}(\left|\frac{31}{5}\right|+\left|\frac{37}{5}\right| + \left|\frac{12}{5}\right|)\}.$$

After simplification, we have

$$\left\|(\frac{1}{3}S_1)\right\|^1_\infty = \ max\,\{\frac{1}{48}(16),\frac{1}{48}(16),\frac{1}{48}(16)\}.$$

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Which implies

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_{\infty}^{1} = max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} < 1.$$

This shows that the scheme S_1 is contractive.

Hence the scheme presented in (9) is C^0 is continuous.

Now we solve for C^1 continuity, multiplying equation (3.9) with $\left(\frac{3z^2}{1+z+z^2}\right)$ and get,

$$M^2_{\frac{3}{10}}(z) = \left(\frac{3z^2}{1+z+z^2}\right)M^1_{\frac{1}{2}}(z).$$

where $M^1_{\frac{3}{10}}(z)$ is defined. After putting the value of $M^1_{\frac{3}{10}}(z)$, we get

$$M^2_{\frac{3}{10}}(z) = \left(\frac{3z^2}{1+z+z^2}\right)z^2(1+z+z^2)^2\left\{\frac{1}{5}+\frac{11}{5}z+\frac{19}{5}z^2+\frac{9}{5}z^3+\frac{9}{5}z^4+\frac{19}{5}z^5+\frac{11}{5}z^6 + \frac{1}{5}z^7\right\}.$$

After simplification we have

$$M^2_{\frac{3}{10}}(z) = 3z^4(1+z+z^2)\left\{\frac{1}{5}+\frac{11}{5}z+\frac{19}{5}z^2+\frac{9}{5}z^3+\frac{9}{5}z^4+\frac{19}{5}z^5+\frac{11}{5}z^6+\frac{1}{5}z^7\right\}.$$

By collecting terms of z, we have

$$M^2_{\frac{3}{10}}(z) = 3\left\{\frac{1}{5}+\frac{11}{5}z+\frac{19}{5}z^2+\frac{9}{5}z^3+\frac{9}{5}z^4+\frac{19}{5}z^5+\frac{11}{5}z^6+\frac{1}{5}z^7\right\}.$$

Let T^1 be the mask of scheme S_2 from (7), we have

$$\mathsf{T}^2 = 3\left\{\frac{1}{5},\frac{11}{5},\frac{19}{5},\frac{9}{5},\frac{9}{5},\frac{19}{5},\frac{11}{5},\frac{1}{5}\right\}. \quad (74)$$

For C^1 continuity then S_2 be the subdivision scheme corresponding to S_2 .

By $n = 2$ and $L = 1$ in (45), we have

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_{\infty}^{1} = max\left\{\frac{1}{3}\sum_j \left|\alpha_{3j}^{[2,1]}\right|,\ \frac{1}{3}\sum_j \left|\alpha_{3j+1}^{[2,1]}\right|,\ \frac{1}{3}\sum_j \left|\alpha_{3j+2}^{[2,1]}\right|\right\}. \quad (75)$$

$$\left\|\left(\frac{1}{3}S_2\right)\right\|_{\infty}^{1} = max\ 3\left\{\frac{1}{48}\left(\left|\frac{1}{5}\right|+\left|\frac{9}{5}\right|+\left|\frac{11}{5}\right|\right),\frac{1}{48}\left(\left|\frac{11}{5}\right|+\left|\frac{9}{5}\right|+\left|\frac{1}{5}\right|\right),\frac{1}{48}\left(\left|\frac{19}{5}\right|+\left|\frac{19}{5}\right|\right)\right\}.$$

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_{\infty}^{1} = max\left\{\frac{1}{16}\left|\frac{21}{5}\right|,\frac{1}{16}\left|\frac{21}{5}\right|,\frac{1}{16}\left|\frac{38}{5}\right|\right\} < 1.$$

Which implies

$$\left\|\left(\frac{1}{3}S_1\right)\right\|_\infty^1 = max\left\{\frac{21}{80},\frac{21}{80},\frac{19}{40}\right\} < 1. \quad (76)$$

This show that scheme S_2 is contractive,

Hence the scheme presented in (9) is C^1 is continuous.

Table 2: *Continuity of our Scheme at different values of parameter.*

β	Continuity
$\frac{1}{2}$	C^1
$\frac{1}{3}$	C^1
$\frac{1}{4}$	C^1
$\frac{1}{5}$	C^1
$\frac{3}{10}$	C^1

6. Parametrization at M_β

The Laurent Polynomial of the 4-point ternary subdivision scheme, is given by

$$M_\beta(z) = \frac{1}{48}(1+z+z^2)^2\{(2-6\beta)z^0 + (7-16\beta)z^1 + (5-4\beta)z^2 + (-6+26\beta)z^3 + (-6+26\beta)z^4 + (5-4\beta)z^5 + (7-16\beta)z^6 + (2-6\beta)z^7\}. \quad (77)$$

We have to find

$\tau = \frac{M'_\beta(1)}{m}$ and $t_j^i = \tau + \frac{(i-\tau)}{m^j}$ By taking the derivative of $M_\beta(z)$, we get

$$(M_\beta)'_1(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\{(2-6\beta)z^0 + (7-16\beta)z^1 + (5-4\beta)z^2 + (-6+26\beta)z^3 + (-6+26\beta)z^4 + (5-4\beta)z^5 + (7-16\beta)z^6 + (2-6\beta)z^7 + \frac{1}{48}(1+z+z^2)^2\{(7-16\beta) + 2(5-4\beta)z^1 + 3(-6+26\beta)z^2 + 4(-6+26\beta)z^3 + 5(5-4\beta)z^4 + 6(7-$$

$16\beta)z^5 + 7(2-6\beta)z^6\}.$ (78)

By substituting $z=1$, we get

$$M'_\beta(1) = \frac{33}{2}.$$

So that

$$\tau = \frac{M'_\beta(1)}{m} = \frac{33}{6}.$$

And

$$t_j^i = \tau + \frac{(i-\tau)}{m^j}.$$

$$t_j^i = \frac{33}{6} + \frac{(i-(\frac{33}{6}))}{3^j}.$$

Hence the 4-point relaxed ternary subdivision scheme at M_β is dual.

Parametrization at $M_{\frac{1}{2}}$

The Laurent Polynomial of the 4-point ternary subdivision scheme at $\beta = \frac{1}{2}$, is given by

$$M_{\frac{1}{2}}(z) = \frac{1}{48}(1+z+z^2)^2\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\}.$$

We have to find

$\tau = \frac{M'_{\frac{1}{2}}(1)}{m}$ and $t_j^i = \tau + \frac{(i-\tau)}{m^j}$ By taking the derivative of $M_{\frac{1}{2}}(z)$, we get

$(M_{\frac{1}{2}})'_1(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\} + \frac{1}{48}(1+$
$z+z^2)^2\{-1+6z+21z^2+28z^3+15z^4-6z^5-7z^6\}.$ (79)

By substituting $z=1$, we get

$$M'_{\frac{1}{2}}(1) = \frac{33}{2}.$$

So that

$$\tau = \frac{M'_{\frac{1}{2}}(1)}{m} = \frac{33}{6}.$$

And

$$t_j^i = \tau + \frac{(i-\tau)}{m^j}.$$

$$t_j^i = \frac{33}{6} + \frac{(i - (\frac{33}{6}))}{3^j}.$$

Hence the 4-point relaxed ternary subdivision scheme at $\beta = \frac{1}{2}$ is dual.

Parametrization at $M_{\frac{1}{3}}$

The Laurent Polynomial of the 4-point ternary subdivision scheme at $\beta = \frac{1}{3}$, is given by

$$M_{\frac{1}{3}}(z) = \frac{1}{48}(1+z+z^2)^2 \frac{1}{3}\{5z + 11z^2 + 8z^3 + 8z^4 + 11z^5 + 5z^6\}.$$

We have to find

$\tau = \frac{M'_{\frac{1}{3}}(1)}{m}$ and $t_j^i = \tau + \frac{(i-\tau)}{m^j}$ By taking the derivative of $M_{\frac{1}{2}}(z)$, we get

$(M_{\frac{1}{3}})'_1(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\frac{1}{3}\{5z + 11z^2 + 8z^3 + 8z^4 + 11z^5 + 5z^6\} + \frac{1}{48}(1+z+$ $z^2)^2\frac{1}{3}\{5 + 22z + 24z^2 + 32z^3 + 55z^4 + 30z^5\}.$ (80)

By substituting $z = 1$, we get

$$M'_{\frac{1}{3}}(1) = \frac{33}{2}.$$

So that

$$\tau = \frac{M'_{\frac{1}{3}}(1)}{m} = \frac{33}{6}.$$

And

$$t_j^i = \tau + \frac{(i-\tau)}{m^j}.$$

$$t_j^i = \frac{33}{6} + \frac{(i - (\frac{33}{6}))}{3^j}.$$

Hence the 4-point relaxed ternary subdivision scheme at $\beta = \frac{1}{3}$ is dual

Parametrization at $M_{\frac{1}{4}}$

The Laurent Polynomial of the 4-point ternary subdivision scheme at $\beta = \frac{1}{4}$, is given by

$$M_{\frac{1}{4}}(z) = \frac{1}{48}(1+z+z^2)^2 \frac{1}{2}\{1+6z+8z^2+z^3+z^4+8z^5+6z^6+z^7\}.$$

We have to find

$\tau = \frac{M'_{\frac{1}{4}}(1)}{m}$ and $t_j^i = \tau + \frac{(i-\tau)}{m^j}$ By taking the derivative of $M_{\frac{1}{4}}(z)$, we get

$$(M_{\frac{1}{4}})'_1(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\frac{1}{2}\{1+6z+8z^2+z^3+z^4+8z^5+6z^6+z^7\} + \frac{1}{48}(1+z+z^2)^2\frac{1}{2}\{6+16z+3z^2+4z^3+40z^4+36z^5+7z^6\}. \quad (81)$$

By substituting $z = 1$, we get

$$M'_{\frac{1}{4}}(1) = \frac{33}{2}.$$

So that

$$\tau = \frac{M'_{\frac{1}{4}}(1)}{m} = \frac{33}{6}.$$

And

$$t_j^i = \tau + \frac{(i-\tau)}{m^j}.$$

$$t_j^i = \frac{33}{6} + \frac{(i-(\frac{33}{6}))}{3^j}.$$

Hence the 4-point relaxed ternary subdivision scheme at $\beta = \frac{1}{4}$ is dual.

Parametrization at $M_{\frac{1}{5}}$

The Laurent Polynomial of the 4-point ternary subdivision scheme at $\beta = \frac{1}{5}$, is given by

$$M_{\frac{1}{5}}(z) = \frac{1}{48}(1+z+z^2)^2 \frac{1}{5}\{4+19z+21z^2-4z^3-4z^4+21z^5+19z^6+4z^7\}.$$

We have to find

$\tau = \frac{M'_{\frac{1}{5}}(1)}{m}$ and $t_j^i = \tau + \frac{(i-\tau)}{m^j}$ By taking the derivative of $M_{\frac{1}{2}}(z)$, we get

$(M_{\frac{1}{5}})'_1(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\frac{1}{5}\{4+19z+21z^2-4z^3-4z^4+21z^5+19z^6+$

$4z^7\} + \frac{1}{48}(1+z+z^2)^2\frac{1}{5}\{19 + 42z - 12z^2 - 16z^3 + 105z^4 + 114z^5 + 28z^6\}.$ (82)

By substituting $z = 1$, we get

$$M'_{\frac{1}{5}}(1) = \frac{33}{2}.$$

So that

$$\tau = \frac{M'_{\frac{1}{5}}(1)}{m} = \frac{33}{6}.$$

And

$$t_j^i = \tau + \frac{(i-\tau)}{m^j}.$$

$$t_j^i = \frac{33}{6} + \frac{(i-(\frac{33}{6}))}{3^j}.$$

Hence the 4-point relaxed ternary subdivision scheme at $\beta = \frac{1}{5}$ is dual.

Parametrization at $M_{\frac{3}{10}}$

The Laurent Polynomial of the 4-point ternary subdivision scheme at $\beta = \frac{3}{10}$, is given by

$$M_{\frac{3}{10}}(z) = \frac{1}{48}(1+z+z^2)^2\frac{1}{5}\{1 + 11z + 19z^2 + 9z^3 + 9z^4 + 19z^5 + 11z^6 + z^7\}.$$

We have to find

$\tau = \frac{M'_{\frac{3}{10}}(1)}{m}$ and $t_j^i = \tau + \frac{(i-\tau)}{m^j}$ By taking the derivative of $M_{\frac{1}{2}}(z)$, we get

$(M_{\frac{3}{10}})'_1(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\frac{1}{5}\{1 + 11z + 19z^2 + 9z^3 + 9z^4 + 19z^5 + 11z^6 +$

$z^7\} + \frac{1}{48}(1+z+z^2)^2\frac{1}{5}\{11 + 38z + 27z^2 + 36z^3 + 95z^4 + 66z^5 + 7z^6\}.$ (83)

By substituting $z = 1$, we get

$$M'_{\frac{3}{10}}(1) = \frac{33}{2}.$$

So that

$$\tau = \frac{M'_{\frac{3}{10}}(1)}{m} = \frac{33}{6}.$$

And

$$t_j^i = \tau + \frac{(i - \tau)}{m^j}.$$

$$t_j^i = \frac{33}{6} + \frac{(i - (\frac{33}{6}))}{3^j}.$$

Hence the 4-point relaxed ternary subdivision scheme at $\beta = \frac{3}{10}$ is dual.

Table 3: *Parametrization of our Scheme at different values of parameter.*

β	Parametrization
$\frac{1}{2}$	$Dual$
$\frac{1}{3}$	$Dual$
$\frac{1}{4}$	$Dual$
$\frac{1}{5}$	$Dual$
$\frac{3}{10}$	$Dual$

7. Degree of polynomial generation at M_β

Theorem 13: Degree of polynomial generation of 4-point ternary scheme.

Proof. the Laurent polynomial the scheme $(M_\beta)_1(z)$

$M_\beta(z) = \frac{1}{48}(1 + z + z^2)^2\{(2 - 6\beta)z^0 + (7 - 16\beta)z^1 + (5 - 4\beta)z^2 + (-6 + 26\beta)z^3 +$

$(-6+26\beta)z^4+(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}.$ (84)

By taking its maximum common factors, we have

$$(M_\beta)_1(z)=(1+z+z^2)^{1+1}b(z)$$

where

$$b(z)=\frac{1}{48}\{(2-6\beta)z^0+(7-16\beta)z^1+(5-4\beta)z^2+(-6+26\beta)z^3+(-6+26\beta)z^4+(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}.$$

So the degree of polynomial generation is 1.

Degree of polynomial generation at $M_{\frac{1}{2}}$

Theorem 14: Degree of polynomial generation of 4-point ternary scheme.

Proof. the Laurent polynomial the scheme $(M_{\frac{1}{2}})_1(z)$

$$M_{\frac{1}{2}}(z)=\frac{1}{48}(1+z+z^2)^2\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\}.$$

By taking its maximum common factors, we have

$$(M_{\frac{1}{2}})_1(z)=(1+z+z^2)^{1+1}b(z)$$

where

$$b(z)=\frac{1}{48}\{-1-z+3z^2+7z^3+7z^4+3z^5-z^6-z^7\}.$$

So the degree of polynomial generation is 1.

Degree of polynomial generation at $M_{\frac{1}{3}}$

Theorem 15: Degree of polynomial generation of 4-point ternary scheme.

Proof. the Laurent polynomial the scheme $(M_{\frac{1}{3}})_1(z)$

$$M_{\frac{1}{3}}(z)=\frac{1}{48}(1+z+z^2)^2\frac{1}{3}\{5z+11z^2+8z^3+8z^4+11z^5+5z^6\}.$$

By taking its maximum common factors, we have

$$(M_{\frac{1}{3}})_1(z)=(1+z+z^2)^{1+1}b(z)$$

where

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$$b(z)=\frac{1}{144}\{5z+11z^2+8z^3+8z^4+11z^5+5z^6\}.$$

So the degree of polynomial generation is 1.

Degree of polynomial generation at $M_{\frac{1}{4}}$

Theorem 16: Degree of polynomial generation of 4-point ternary scheme.

Proof. the Laurent polynomial the scheme $(M_{\frac{1}{4}})_1(z)$

$$M_{\frac{1}{4}}(z)=\frac{1}{48}(1+z+z^2)^2\frac{1}{2}\{1+6z+8z^2+z^3+z^4+8z^5+6z^6+z^7\}.$$

By taking its maximum common factors, we have

$$(M_{\frac{1}{4}})_1(z)=(1+z+z^2)^{1+1}b(z)$$

where

$$b(z)=\frac{1}{96}\{1+6z+8z^2+z^3+z^4+8z^5+6z^6+z^7\}.$$

So the degree of polynomial generation is 1.

Degree of polynomial generation at $M_{\frac{1}{5}}$

Theorem 17: Degree of polynomial generation of 4-point ternary scheme.

Proof. the Laurent polynomial the scheme $(M_{\frac{1}{5}})_1(z)$

$$M_{\frac{1}{5}}(z)=\frac{1}{48}(1+z+z^2)^2\frac{1}{5}\{4+19z+21z^2-4z^3-4z^4+21z^5+19z^6+4z^7\}.$$

By taking its maximum common factors, we have

$$(M_{\frac{1}{5}})_1(z)=(1+z+z^2)^{1+1}b(z)$$

where

$$b(z)=\frac{1}{240}\{4+19z+21z^2-4z^3-4z^4+21z^5+19z^6+4z^7\}.$$

So the degree of polynomial generation is 1.

Degree of polynomial generation at $M_{\frac{3}{10}}$

Theorem 18: Degree of polynomial generation of 4-point ternary scheme.

Proof. the Laurent polynomial the scheme $(M_{\frac{3}{10}})_1(z)$

$$M_{\frac{3}{10}}(z) = \frac{1}{48}(1+z+z^2)^2\frac{1}{5}\{1+11z+19z^2+9z^3+9z^4+19z^5+11z^6+z^7\}.$$

By taking its maximum common factors, we have

$$(M_{\frac{3}{10}})_1(z) = (1+z+z^2)^{1+1}b(z)$$

where

$$b(z) = \frac{1}{240}\{1+11z+19z^2+9z^3+9z^4+19z^5+11z^6+z^7\}.$$

So the degree of polynomial generation is 1.

Table 4: *Degree of generation of our Scheme at different values of parameter.*

β	Degree of generation
$\frac{1}{2}$	1
$\frac{1}{3}$	1
$\frac{1}{4}$	1
$\frac{1}{5}$	1
$\frac{3}{10}$	1

8. Degree of polynomial reproduction of 4-point ternary subdivision scheme.

In this section, we will discuss the ability of polynomial reproduction of 3-point quaternary subdivision scheme.

Degree of polynomial reproduction at M_β

Theorem 19: A convergent subdivision scheme reproduced polynomial of degree 'd' w.r.t parametrization with $\tau = \frac{M'_\beta(1)}{m}$, iff, it satisfies the two conditions.

$$(M_\beta)^k(1) = m\prod_{l=0}^{k-1}(\tau - \ell).$$

And

$$(M_\beta)^k(a_m^j) = 0.$$

Where

$$a_m^j = e^{\frac{2\pi ip}{m}}$$

Proof. The Laurent polynomial of 4-point of ternary subdivision scheme is.

$$M_\beta(z) = \frac{1}{48}(1+z+z^2)^2\{(2-6\beta)z^0 + (7-16\beta)z^1 + (5-4\beta)z^2 + (-6+26\beta)z^3 + (-6+26\beta)z^4 + (5-4\beta)z^5 + (7-16\beta)z^6 + (2-6\beta)z^7\}. \quad (85)$$

By substituting $z = 1$ we get $M_\beta(1) = 3$

$$(M_\beta)^k(1) = m\prod_{\ell=0}^{k-1}(\tau - \ell).$$

For $k = 0, \ell = 0$ and $m = 3$

$$(M_\beta)^0(1) = 3\prod_{0=0}^{0-1}(\frac{33}{6} - 0).$$

After simplifying, we get $3 = 3$ (satisfied) And

$$(M_\beta)^k(a_m^j) = 0.$$

For $j = 1, k = 0$ and $m = 3$

$$(M_\beta)^0(a_3^1) = 0.$$

$$M_\beta(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 0$ and $m = 3$

$$(M_\beta)^0(a_3^2) = 0.$$

$$M_\beta(e^{\pi\ell}) = 0.$$

For $j=3, k=0$ and $m=3$

$$(M_\beta)^0(a_3^3)=0.$$

$$M_\beta(e^{\frac{3\pi\ell}{3}})=0.$$

$(M_\beta)'_1(z)=\frac{1}{32}2(1+z+z^2)(1+2z)\{(2-6\beta)z^0+(7-16\beta)z^1+(5-4\beta)z^2+(-6+26\beta)z^3+(-6+26\beta)z^4+(5-4\beta)z^5+(7-16\beta)z^6+(2-6\beta)z^7\}+\frac{1}{32}(1+z+z^2)^2\{(7-16\beta)+2(5-4\beta)z^1+3(-6+26\beta)z^2+4(-6+26\beta)z^3+5(5-4\beta)z^4+6(7-16\beta)z^5+7(2-6\beta)z^6\}.$ (86)

By substituting $z=1$ we get $M'_\beta(1)=\frac{33}{2}$

We know that, $\tau=\frac{M'_\beta(1)}{m}$

Substituting the values of m and $M'_\beta(1)$, we have

$$\tau=\frac{33}{6}$$

$$(M_\beta)^k(1)=m\prod_{\ell=0}^{k-1}(\tau-\ell).$$

For $k=1, \ell=0$ and $m=3$

$$M'_\beta(1)=3\prod_{0=0}^{1-1}(\frac{33}{6}-0).$$

$$\frac{33}{2}=3(\frac{33}{6})$$

After simplifying, we get $\frac{33}{2}=\frac{33}{2}$ (satisfied)

And

$$(M_\beta)^k(a_m^j)=0.$$

For $j=1, k=1$ and $m=3$

$$M'_\beta(a_3^1)=0.$$

$$M'_\beta(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 1$ and $m = 3$

$$M'_\beta(a_3^2) = 0.$$

$$M'_\beta(e^{\pi\ell}) = 0.$$

For $j = 3, k = 1$ and $m = 3$

$$M'_\beta(a_3^3) = 0.$$

$$M'_\beta(e^{\frac{3\pi\ell}{3}}) = 0.$$

Degree of polynomial reproduction at $M_{\frac{1}{2}}$

Theorem 20: A convergent subdivision scheme reproduced polynomial of degree 'd' w.r.t parametrization with $\tau = \frac{M'_{\frac{1}{2}}(1)}{m}$, iff, it satisfies the two conditions.

$$(M_{\frac{1}{2}})^k(1) = m\prod_{l=0}^{k-1}(\tau - \ell).$$

And

$$(M_{\frac{1}{2}})^k(a_m^j) = 0.$$

Where

$$a_m^j = e^{\frac{2\pi ip}{m}}$$

Proof. The Laurent polynomial of 4-point of ternary subdivision scheme is.

$$M_{\frac{1}{2}}(z) = \frac{1}{48}(1 + z + z^2)^2\{-1 - z + 3z^2 + 7z^3 + 7z^4 + 3z^5 - z^6 - z^7\}.$$

By substituting $z = 1$ we get $M_{\frac{1}{2}}(1) = 3$

$$(M_{\frac{1}{2}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau - \ell).$$

For $k = 0, \ell = 0$ and $m = 3$

$$(M_{\frac{1}{2}})^0(1) = 3\prod_{0=0}^{0-1}(\frac{33}{6} - 0).$$

After simplifying, we get $3 = 3$ (satisfied) And

$$(M_{\frac{1}{2}})^k(a_m^j) = 0.$$

For $j = 1, k = 0$ and $m = 3$

$$(M_{\frac{1}{2}})^0(a_3^1) = 0.$$

$$M_{\frac{1}{2}}(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 0$ and $m = 3$

$$(M_{\frac{1}{2}})^0(a_3^2) = 0.$$

$$M_{\frac{1}{2}}(e^{\pi\ell}) = 0.$$

For $j = 3, k = 0$ and $m = 3$

$$(M_{\frac{1}{2}})^0(a_3^3) = 0.$$

$$M_{\frac{1}{2}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

$$(M_{\frac{1}{2}})_1'(z) = \frac{1}{48}2(1 + z + z^2)(1 + 2z)\{-1 - z + 3z^2 + 7z^3 + 7z^4 + 3z^5 - z^6 - z^7\} + \frac{1}{48}(1 + z + z^2)^2\{-1 + 6z + 21z^2 + 28z^3 + 15z^4 - 6z^5 - 7z^6\}. \quad (87)$$

By substituting $z = 1$ we get $M_{\frac{1}{2}}'(1) = \frac{33}{2}$

We know that, $\tau = \frac{M_{\frac{1}{2}}'(1)}{m}$

Substituting the values of m and $M_{\frac{1}{2}}'(1)$, we have

$$\tau = \frac{33}{6}$$

$$(M_{\frac{1}{2}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau - \ell).$$

For $k = 1, \ell = 0$ and $m = 3$

$$M'_{\frac{1}{2}}(1) = 3\prod_{0=0}^{1-1}(\frac{33}{6} - 0).$$

$$\frac{33}{2} = 3(\frac{33}{6})$$

After simplifying, we get $\frac{33}{2} = \frac{33}{2}$ (satisfied)

And

$$(M_{\frac{1}{2}})^k(a_m^j) = 0.$$

For $j = 1, k = 1$ and $m = 3$

$$M'_{\frac{1}{2}}(a_3^1) = 0.$$

$$M'_{\frac{1}{2}}(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 1$ and $m = 3$

$$M'_{\frac{1}{2}}(a_3^2) = 0.$$

$$M'_{\frac{1}{2}}(e^{\pi\ell}) = 0.$$

For $j = 3, k = 1$ and $m = 3$

$$M'_{\frac{1}{2}}(a_3^3) = 0.$$

$$M'_{\frac{1}{2}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

Degree of polynomial reproduction at $M_{\frac{1}{3}}$

Theorem 21: A convergent subdivision scheme reproduced polynomial of degree 'd' w.r.t parametrization with $\tau = \frac{M'_{\frac{1}{3}}(1)}{m}$, iff, it satisfies the two conditions.

$$(M_{\frac{1}{3}})^k(1) = m\prod_{l=0}^{k-1}(\tau - \ell).$$

And

$$(M_{\frac{1}{3}})^k(a_m^j) = 0.$$

Where

$$a_m^j = e^{\frac{2\pi i p}{m}}$$

Proof. The Laurent polynomial of 4-point of ternary subdivision scheme is.

$$M_{\frac{1}{3}}(z) = \frac{1}{48}(1+z+z^2)^2\frac{1}{3}\{5z+11z^2+8z^3+8z^4+11z^5+5z^6\}.$$

By substituting $z = 1$ we get $M_{\frac{1}{3}}(1) = 3$

$$(M_{\frac{1}{3}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau-\ell).$$

For $k = 0, \ell = 0$ and $m = 3$

$$(M_{\frac{1}{3}})^0(1) = 3\prod_{0=0}^{0-1}(\frac{33}{6}-0).$$

After simplifying, we get $3 = 3$ (satisfied) And

$$(M_{\frac{1}{3}})^k(a_m^j) = 0.$$

For $j = 1, k = 0$ and $m = 3$

$$(M_{\frac{1}{3}})^0(a_3^1) = 0.$$

$$M_{\frac{1}{3}}(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 0$ and $m = 3$

$$(M_{\frac{1}{3}})^0(a_3^2) = 0.$$

$$M_{\frac{1}{3}}(e^{\pi\ell}) = 0.$$

For $j = 3, k = 0$ and $m = 3$

$$(M_{\frac{1}{3}})^0(a_3^3) = 0.$$

$$M_{\frac{1}{3}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

$$(M_{\frac{1}{3}})_1'(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\frac{1}{3}\{5z+11z^2+8z^3+8z^4+11z^5+5z^6\}+\frac{1}{48}(1+$$

$z + z^2)^2 \frac{1}{3}\{5 + 22z + 24z^2 + 32z^3 + 55z^4 + 30z^5\}.$ (88)

By substituting $z = 1$ we get $M'_{\frac{1}{3}}(1) = \frac{33}{2}$

We know that, $\tau = \frac{M'_{\frac{1}{3}}(1)}{m}$

Substituting the values of m and $M'_{\frac{1}{3}}(1)$, we have

$$\tau = \frac{33}{6}$$

$$(M_{\frac{1}{3}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau - \ell).$$

For $k = 1, \ell = 0$ and $m = 3$

$$M'_{\frac{1}{3}}(1) = 3\prod_{0=0}^{1-1}(\frac{33}{6} - 0).$$

$$\frac{33}{2} = 3(\frac{33}{6})$$

After simplifying, we get $\frac{33}{2} = \frac{33}{2}$ (satisfied)

And

$$(M_{\frac{1}{3}})^k(a_m^j) = 0.$$

For $j = 1, k = 1$ and $m = 3$

$$M'_{\frac{1}{3}}(a_3^1) = 0.$$

$$M'_{\frac{1}{3}}(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 1$ and $m = 3$

$$M'_{\frac{1}{3}}(a_3^2) = 0.$$

$$M'_{\frac{1}{3}}(e^{\pi\ell}) = 0.$$

For $j = 3, k = 1$ and $m = 3$

$$M'_{\frac{1}{3}}(a_3^3) = 0.$$

$$M'_{\frac{1}{3}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

Degree of polynomial reproduction at $M_{\frac{1}{4}}$

Theorem 22: A convergent subdivision scheme reproduced polynomial of degree 'd' w.r.t parametrization with $\tau = \frac{M'_{\frac{1}{4}}(1)}{m}$, iff, it satisfies the two conditions.

$$(M_{\frac{1}{4}})^k(1) = m\prod_{l=0}^{k-1}(\tau - \ell).$$

And

$$(M_{\frac{1}{4}})^k(a_m^j) = 0.$$

Where

$$a_m^j = e^{\frac{2\pi ip}{m}}$$

Proof. The Laurent polynomial of 4-point of ternary subdivision scheme is.

$$M_{\frac{1}{4}}(z) = \frac{1}{48}(1+z+z^2)^2\frac{1}{2}\{1+6z+8z^2+z^3+z^4+8z^5+6z^6+z^7\}.$$

By substituting $z = 1$ we get $M_{\frac{1}{4}}(1) = 3$

$$(M_{\frac{1}{4}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau - \ell).$$

For $k = 0, \ell = 0$ and $m = 3$

$$(M_{\frac{1}{4}})^0(1) = 3\prod_{0=0}^{0-1}(\frac{33}{6} - 0).$$

After simplifying, we get $3 = 3$ (satisfied) And

$$(M_{\frac{1}{4}})^k(a_m^j) = 0.$$

For $j = 1, k = 0$ and $m = 3$

$$(M_{\frac{1}{4}})^0(a_3^1) = 0.$$

$$M_{\frac{1}{4}}(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 0$ and $m = 3$

$$(M_{\frac{1}{4}})^0(a_3^2) = 0.$$

$$M_{\frac{1}{4}}(e^{\pi\ell}) = 0.$$

For $j = 3, k = 0$ and $m = 3$

$$(M_{\frac{1}{4}})^0(a_3^3) = 0.$$

$$M_{\frac{1}{4}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

$(M_{\frac{1}{4}})'_1(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\frac{1}{2}\{1+6z+8z^2+z^3+z^4+8z^5+6z^6+z^7\} + \frac{1}{48}(1+z+z^2)^2\frac{1}{2}\{6+16z+3z^2+4z^3+40z^4+36z^5+7z^6\}.$ (89)

By substituting $z = 1$ we get $M'_{\frac{1}{4}}(1) = \frac{33}{2}$

We know that, $\tau = \frac{M'_{\frac{1}{4}}(1)}{m}$

Substituting the values of m and $M'_{\frac{1}{4}}(1)$, we have

$$\tau = \frac{33}{6}$$

$$(M_{\frac{1}{4}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau - \ell).$$

For $k = 1, \ell = 0$ and $m = 3$

$$M'_{\frac{1}{4}}(1) = 3\prod_{0=0}^{1-1}(\frac{33}{6} - 0).$$

$$\frac{33}{2} = 3(\frac{33}{6})$$

After simplifying, we get $\frac{33}{2} = \frac{33}{2}$ (satisfied)

And

$$(M_{\frac{1}{4}})^k(a_m^j) = 0.$$

For $j = 1, k = 1$ and $m = 3$

$$M'_{\frac{1}{4}}(a_3^1) = 0.$$

$$M'_{\frac{1}{4}}(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 1$ and $m = 3$

$$M'_{\frac{1}{4}}(a_3^2) = 0.$$

$$M'_{\frac{1}{4}}(e^{\pi\ell}) = 0.$$

For $j = 3, k = 1$ and $m = 3$

$$M'_{\frac{1}{4}}(a_3^3) = 0.$$

$$M'_{\frac{1}{4}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

Degree of polynomial reproduction at $M_{\frac{1}{5}}$

Theorem 23: A convergent subdivision scheme reproduced polynomial of degree 'd' w.r.t parametrization with $\tau = \frac{M'_{\frac{1}{5}}(1)}{m}$, iff, it satisfies the two conditions.

$$(M_{\frac{1}{5}})^k(1) = m\prod_{l=0}^{k-1}(\tau - \ell).$$

And

$$(M_{\frac{1}{5}})^k(a_m^j) = 0.$$

Where

$$a_m^j = e^{\frac{2\pi ip}{m}}$$

Proof. The Laurent polynomial of 4-point of ternary subdivision scheme is.

$$M_{\frac{1}{5}}(z) = \frac{1}{48}(1 + z + z^2)^2\frac{1}{5}\{4 + 19z + 21z^2 - 4z^3 - 4z^4 + 21z^5 + 19z^6 + 4z^7\}.$$

By substituting $z = 1$ we get $M_{\frac{1}{5}}(1) = 3$

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$$(M_{\frac{1}{5}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau-\ell).$$

For $k=0, \ell=0$ and $m=3$

$$(M_{\frac{1}{5}})^0(1) = 3\prod_{0=0}^{0-1}(\frac{33}{6}-0).$$

After simplifying, we get $3=3$ (satisfied) And

$$(M_{\frac{1}{5}})^k(a_m^j) = 0.$$

For $j=1, k=0$ and $m=3$

$$(M_{\frac{1}{5}})^0(a_3^1) = 0.$$

$$M_{\frac{1}{5}}(e^{\frac{\pi}{2}}) = 0.$$

For $j=2, k=0$ and $m=3$

$$(M_{\frac{1}{5}})^0(a_3^2) = 0.$$

$$M_{\frac{1}{5}}(e^{\pi\ell}) = 0.$$

For $j=3, k=0$ and $m=3$

$$(M_{\frac{1}{5}})^0(a_3^3) = 0.$$

$$M_{\frac{1}{5}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

$$(M_{\frac{1}{5}})_1'(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\frac{1}{5}\{4+19z+21z^2-4z^3-4z^4+21z^5+19z^6+4z^7\}+\frac{1}{48}(1+z+z^2)^2\frac{1}{5}\{19+42z-12z^2-16z^3+105z^4+114z^5+28z^6\}. \quad (90)$$

By substituting $z=1$ we get $M_{\frac{1}{5}}'(1) = \frac{33}{2}$

We know that, $\tau = \frac{M_{\frac{1}{5}}'(1)}{m}$

Substituting the values of m and $M_{\frac{1}{5}}'(1)$, we have

$$\tau = \frac{33}{6}$$

$$(M_{\frac{1}{5}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau-\ell).$$

For $k=1, \ell=0$ and $m=3$

$$M'_{\frac{1}{5}}(1) = 3\prod_{0=0}^{1-1}(\frac{33}{6}-0).$$

$$\frac{33}{2} = 3(\frac{33}{6})$$

After simplifying, we get $\frac{33}{2} = \frac{33}{2}$ (satisfied)

And

$$(M_{\frac{1}{5}})^k(a_m^j) = 0.$$

For $j=1, k=1$ and $m=3$

$$M'_{\frac{1}{5}}(a_3^1) = 0.$$

$$M'_{\frac{1}{5}}(e^{\frac{\pi}{2}}) = 0.$$

For $j=2, k=1$ and $m=3$

$$M'_{\frac{1}{5}}(a_3^2) = 0.$$

$$M'_{\frac{1}{5}}(e^{\pi\ell}) = 0.$$

For $j=3, k=1$ and $m=3$

$$M'_{\frac{1}{5}}(a_3^3) = 0.$$

$$M'_{\frac{1}{5}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

Degree of polynomial reproduction at $M_{\frac{3}{10}}$

Theorem 24: A convergent subdivision scheme reproduced polynomial of degree 'd' w.r.t parametrization with $\tau = \frac{M'_{\frac{3}{10}}(1)}{m}$, iff, it satisfies the two conditions.

$$(M_{\frac{3}{10}})^k(1) = m\prod_{l=0}^{k-1}(\tau - \ell).$$

And

$$(M_{\frac{3}{10}})^k(a_m^j) = 0.$$

Where

$$a_m^j = e^{\frac{2\pi ip}{m}}$$

Proof. The Laurent polynomial of 4-point of ternary subdivision scheme is.

$$M_{\frac{3}{10}}(z) = \frac{1}{48}(1+z+z^2)^2\frac{1}{5}\{1+11z+19z^2+9z^3+9z^4+19z^5+11z^6+z^7\}.$$

By substituting $z = 1$ we get $M_{\frac{3}{10}}(1) = 3$

$$(M_{\frac{3}{10}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau - \ell).$$

For $k = 0, \ell = 0$ and $m = 3$

$$(M_{\frac{3}{10}})^0(1) = 3\prod_{0=0}^{0-1}(\frac{33}{6} - 0).$$

After simplifying, we get $3 = 3$ (satisfied) And

$$(M_{\frac{3}{10}})^k(a_m^j) = 0.$$

For $j = 1, k = 0$ and $m = 3$

$$(M_{\frac{3}{10}})^0(a_3^1) = 0.$$

$$M_{\frac{3}{10}}(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 0$ and $m = 3$

$$(M_{\frac{3}{10}})^0(a_3^2) = 0.$$

$$M_{\frac{3}{10}}(e^{\pi\ell}) = 0.$$

For $j = 3, k = 0$ and $m = 3$

$$(M_{\frac{3}{10}})^0(a_3^3) = 0.$$

$$M_{\frac{3}{10}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

$(M_{\frac{3}{10}})'_1(z) = \frac{1}{48}2(1+z+z^2)(1+2z)\frac{1}{5}\{1+11z+19z^2+9z^3+9z^4+19z^5+11z^6+ z^7\} + \frac{1}{48}(1+z+z^2)^2\frac{1}{5}\{11+38z+27z^2+36z^3+95z^4+66z^5+7z^6\}$. (91)

By substituting $z = 1$ we get $M'_{\frac{3}{10}}(1) = \frac{33}{2}$

We know that, $\tau = \frac{M'_{\frac{3}{10}}(1)}{m}$

Substituting the values of m and $M'_{\frac{3}{10}}(1)$, we have $\tau = \frac{33}{6}$

$$(M_{\frac{3}{10}})^k(1) = m\prod_{\ell=0}^{k-1}(\tau - \ell).$$

For $k = 1, \ell = 0$ and $m = 3$

$$M'_{\frac{3}{10}}(1) = 3\prod_{0=0}^{1-1}(\frac{33}{6} - 0).$$

$$\frac{33}{2} = 3(\frac{33}{6})$$

After simplifying, we get $\frac{33}{2} = \frac{33}{2}$ (satisfied)

And

$$(M_{\frac{3}{10}})^k(a_m^j) = 0.$$

For $j = 1, k = 1$ and $m = 3$

$$M'_{\frac{3}{10}}(a_3^1) = 0.$$

$$M'_{\frac{3}{10}}(e^{\frac{\pi}{2}}) = 0.$$

For $j = 2, k = 1$ and $m = 3$

$$M'_{\frac{3}{10}}(a_3^2) = 0.$$

$$M'_{\frac{3}{10}}(e^{\pi\ell}) = 0.$$

For $j = 3, k = 1$ and $m = 3$

$$M'_{\frac{3}{10}}(a_3^3) = 0.$$

$$M'_{\frac{3}{10}}(e^{\frac{3\pi\ell}{3}}) = 0.$$

9. Limit stencils of a 4-point ternary subdivision scheme at M_β

Theorem 25: To find the limit stencil of parametric 4-point ternary subdivision scheme.

Proof. We will put $m = 0$ in the given subdivision scheme and we get,

$$h_{-1}^{t+1} = (\frac{25}{48} - \frac{9\beta}{8})h_{-1}^t + (\frac{-1}{6} + \frac{17\beta}{8})h_0^t + (\frac{29}{48} - \frac{7\beta}{8})h_1^t + (\frac{1}{24} - \frac{1\beta}{8})h_2^t,$$

$$h_0^{t+1} = (\frac{11}{48} - \frac{7\beta}{12})h_{-1}^t + (\frac{13}{48} + \frac{7\beta}{12})h_0^t + (\frac{13}{48} + \frac{7\beta}{12})h_1^t + (\frac{11}{48} - \frac{7\beta}{12})h_2^t,$$

$$h_1^{t+1} = (\frac{1}{24} - \frac{1\beta}{8})h_{-1}^t + (\frac{29}{48} - \frac{7\beta}{8})h_0^t + (\frac{-1}{6} + \frac{17\beta}{8})h_1^t + (\frac{25}{48} - \frac{9\beta}{8})h_2^t.$$

We will put $m = 1$ in the given subdivision scheme and we get,

$$h_2^{t+1} = (\frac{25}{48} - \frac{9\beta}{8})h_0^t + (\frac{-1}{6} + \frac{17\beta}{8})h_1^t + (\frac{29}{48} - \frac{7\beta}{8})h_2^t + (\frac{1}{24} - \frac{1\beta}{8})h_3^t,$$

$$h_3^{t+1} = (\frac{11}{48} - \frac{7\beta}{12})h_0^t + (\frac{13}{48} + \frac{7\beta}{12})h_1^t + (\frac{13}{48} + \frac{7\beta}{12})h_2^t + (\frac{11}{48} - \frac{7\beta}{12})h_3^t,$$

$$h_4^{t+1} = (\frac{1}{24} - \frac{1\beta}{8})h_0^t + (\frac{29}{48} - \frac{7\beta}{8})h_1^t + (\frac{-1}{6} + \frac{17\beta}{8})h_2^t + (\frac{25}{48} - \frac{9\beta}{8})h_3^t.$$

The subdivision matrix of the scheme defined by is

$$\begin{pmatrix} h_0^{t+1} \\ h_1^{t+1} \\ h_2^{t+1} \\ h_3^{t+1} \end{pmatrix} = \begin{pmatrix} \frac{11}{48} - \frac{7\beta}{12} & \frac{13}{48} + \frac{7\beta}{12} & \frac{13}{48} + \frac{7\beta}{12} & \frac{11}{48} - \frac{7\beta}{12} \\ \frac{1}{24} - \frac{1\beta}{8} & \frac{29}{48} - \frac{7\beta}{8} & \frac{-1}{6} + \frac{17\beta}{8} & \frac{25}{48} - \frac{9\beta}{8} \\ \frac{25}{48} - \frac{9\beta}{8} & \frac{-1}{6} + \frac{17\beta}{8} & \frac{29}{48} - \frac{7\beta}{8} & \frac{1}{24} - \frac{1\beta}{8} \\ \frac{11}{48} - \frac{7\beta}{12} & \frac{13}{48} + \frac{7\beta}{12} & \frac{13}{48} + \frac{7\beta}{12} & \frac{11}{48} - \frac{7\beta}{12} \end{pmatrix} \begin{pmatrix} h_0^t \\ h_1^t \\ h_2^t \\ h_3^t \end{pmatrix}$$

It can also be written as.

$$h^{t+1} = Sh^t.$$

Now we find subdivision matrix "S" of given scheme is

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$$S=\begin{pmatrix}\frac{11}{48}-\frac{7\beta}{12} & \frac{13}{48}+\frac{7\beta}{12} & \frac{13}{48}+\frac{7\beta}{12} & \frac{11}{48}-\frac{7\beta}{12}\\ \frac{1}{24}-\frac{1\beta}{8} & \frac{29}{48}-\frac{7\beta}{8} & \frac{-1}{6}+\frac{17\beta}{8} & \frac{25}{48}-\frac{9\beta}{8}\\ \frac{25}{48}-\frac{9\beta}{8} & \frac{-1}{6}+\frac{17\beta}{8} & \frac{29}{48}-\frac{7\beta}{8} & \frac{1}{24}-\frac{1\beta}{8}\\ \frac{11}{48}-\frac{7\beta}{12} & \frac{13}{48}+\frac{7\beta}{12} & \frac{13}{48}+\frac{7\beta}{12} & \frac{11}{48}-\frac{7\beta}{12}\end{pmatrix}.$$

we find eigen value's and eigen vector's of matrix "S" Now we shall compute the eigen decomposition of subdivision scheme. For this

$$\lambda=\begin{pmatrix}0\\1\\-3\beta+\frac{37}{48}\\\frac{\beta}{12}-\frac{5}{48}\end{pmatrix}.$$

The matrix of eigen vector conversing to eigen value is has

$$Q=\begin{pmatrix}0 & 1 & 1 & -1\\ -1 & 1 & \frac{3}{2}\frac{20\beta-9}{13+28\beta} & -\frac{48\beta-23}{144\beta-37}\\ 1 & 1 & \frac{3}{2}\frac{20\beta-9}{13+28\beta} & \frac{48\beta-23}{144\beta-37}\\ 0 & 1 & 1 & 1\end{pmatrix}.$$

As we know that eigen decomposition of S is,

$$S=R\bigtriangleup Q,$$

Let:

$$\bigtriangleup=\begin{pmatrix}0\\1\\-3\beta+\frac{37}{48}\\\frac{\beta}{12}-\frac{5}{48}\end{pmatrix}.$$

Hence the diagonal matrix of eigen value can be written as,

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$$\Delta^{e}=\begin{pmatrix}(0)^{e} & 0 & 0 & 0\\ 0 & (1)^{e} & 0 & 0\\ 0 & 0 & (-3\beta+\frac{37}{48})^{e} & 0\\ 0 & 0 & 0 & (\frac{1\beta}{12}-\frac{5}{48})^{e}\end{pmatrix}.$$

Therefore,

$$\lim_{e\to\infty}(\Delta)^{e}=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Let the matrix of eigen vectors corresponding to the eigen value is,

$$Q=\begin{pmatrix}0 & 1 & 1 & -1\\ -1 & 1 & \frac{3}{2}\frac{20\beta-9}{13+28\beta} & -\frac{48\beta-23}{144\beta-37}\\ 1 & 1 & \frac{3}{2}\frac{20\beta-9}{13+28\beta} & \frac{48\beta-23}{144\beta-37}\\ 0 & 1 & 1 & 1\end{pmatrix}.$$

And inverse matrix Q^{-1} is,

$$Q^{-1}=\begin{pmatrix}\frac{1}{2}\frac{48\beta-23}{144\beta-37} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2}\frac{48\beta-23}{144\beta-37}\\ \frac{3}{2}\frac{20\beta-9}{4\beta-53} & -\frac{13+28\beta}{4\beta-53} & -\frac{13+28\beta}{4\beta-53} & \frac{3}{2}\frac{20\beta-9}{4\beta-53}\\ -\frac{13+28\beta}{4\beta-53} & \frac{13+28\beta}{4\beta-53} & \frac{13+28\beta}{4\beta-53} & -\frac{13+28\beta}{4\beta-53}\\ \frac{-1}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}.$$

Since

$$\lim_{e\to\infty}(\S)^{e}=R\ \ Q\lim_{e\to\infty}(\Delta)^{e},$$

Implies that

$$Q\lim_{e\to\infty}(\Delta)^{e}=\begin{pmatrix}0 & 1 & 1 & -1\\ -1 & 1 & \frac{3}{2}\frac{20\beta-9}{13+28\beta} & -\frac{48\beta-23}{144\beta-37}\\ 1 & 1 & \frac{3}{2}\frac{20\beta-9}{13+28\beta} & \frac{48\beta-23}{144\beta-37}\\ 0 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

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Again implies that

$$Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}0&1&0&0\\0&1&0&0\\0&1&0&0\\0&1&0&0\end{pmatrix}.$$

More implies.

$$Q^{-1}\ \ Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}0&1&0&0\\0&1&0&0\\0&1&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\frac{48\beta-23}{144\beta-37}&\frac{-1}{2}&\frac{1}{2}&\frac{-1}{2}\frac{48\beta-23}{144\beta-37}\\\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\\-\frac{13+28\beta}{4\beta-53}&\frac{13+28\beta}{4\beta-53}&\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}\\\frac{-1}{2}&0&0&\frac{1}{2}\end{pmatrix}$$

$$\begin{pmatrix}h_0^\infty\\h_1^\infty\\h_2^\infty\\h_3^\infty\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\\\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\\\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\\\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\end{pmatrix}=\begin{pmatrix}h_0^0\\h_1^0\\h_2^0\\h_3^0\end{pmatrix}$$

$$\lim_{e\to\infty}(\S)^e=R\ \ Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\\\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\\\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\\\frac{3}{2}\frac{20\beta-9}{4B-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\end{pmatrix}.$$

This implies that limit stencils.

$$\lim_{e\to\infty}(\S)^e=\begin{pmatrix}\frac{3}{2}\frac{20\beta-9}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&-\frac{13+28\beta}{4\beta-53}&\frac{3}{2}\frac{20\beta-9}{4\beta-53}\end{pmatrix}.$$

Limit stencils of a 4-point ternary subdivision scheme at $M_{\frac{1}{2}}$

Theorem 26: To find the limit stencil of parametric 4-point ternary subdivision scheme.

Proof. We will put $m=0$ in the given subdivision scheme and we get,

$$h_{-1}^{t+1}=\frac{-1}{24}h_{-1}^{t}+\frac{43}{48}h_{0}^{t}+\frac{1}{6}h_{1}^{t}+\frac{-1}{48}h_{2}^{t},$$

$$h_{0}^{t+1}=\frac{-1}{16}h_{-1}^{t}+\frac{9}{16}h_{0}^{t}+\frac{9}{16}h_{1}^{t}+\frac{-1}{16}h_{2}^{t},$$

$$h_{1}^{t+1}=\frac{-1}{48}h_{-1}^{t}+\frac{1}{6}h_{0}^{t}+\frac{43}{48}h_{1}^{t}+\frac{-1}{24}h_{2}^{t}.$$

We will put $m=1$ in the given subdivision scheme and we get,

$$h_{2}^{t+1}=\frac{-1}{24}h_{0}^{t}+\frac{43}{48}h_{1}^{t}+\frac{1}{6}h_{2}^{t}+\frac{-1}{48}h_{3}^{t},$$

$$h_{3}^{t+1}=\frac{-1}{16}h_{0}^{t}+\frac{9}{16}h_{1}^{t}+\frac{9}{16}h_{2}^{t}+\frac{-1}{16}h_{3}^{t},$$

$$h_{4}^{t+1}=\frac{-1}{48}h_{0}^{t}+\frac{1}{6}h_{1}^{t}+\frac{43}{48}h_{2}^{t}+\frac{-1}{24}h_{3}^{t}.$$

The subdivision matrix of the scheme defined by is

$$\begin{pmatrix} h_0^{t+1} \\ h_1^{t+1} \\ h_2^{t+1} \\ h_3^{t+1} \end{pmatrix}=\begin{pmatrix} -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \\ -\frac{1}{48} & \frac{1}{6} & \frac{43}{48} & -\frac{1}{24} \\ -\frac{1}{24} & \frac{43}{48} & \frac{1}{6} & -\frac{1}{48} \\ -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \end{pmatrix}\begin{pmatrix} h_0^{t} \\ h_1^{t} \\ h_2^{t} \\ h_3^{t} \end{pmatrix}$$

It can also be written as.

$$h^{t+1}=Sh^{t}.$$

Now we find subdivision matrix "S" of given scheme is

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$$S=\begin{pmatrix}-\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16}\\ -\frac{1}{48} & \frac{1}{6} & \frac{43}{48} & -\frac{1}{24}\\ -\frac{1}{24} & \frac{43}{48} & \frac{1}{6} & -\frac{1}{48}\\ -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16}\end{pmatrix}.$$

we find eigen value's and eigen vector's of matrix "S" Now we shall compute the eigen decomposition of subdivision scheme. For this

$$\lambda=\begin{pmatrix}0\\ 1\\ -\frac{35}{48}\\ -\frac{1}{16}\end{pmatrix}.$$

The matrix of eigen vector conversing to eigen value is has

$$Q=\begin{pmatrix}0 & 1 & 1 & -1\\ -1 & 1 & \frac{1}{18} & -\frac{1}{35}\\ 1 & 1 & \frac{1}{18} & \frac{1}{35}\\ 0 & 1 & 1 & 1\end{pmatrix}.$$

As we know that eigen decomposition of S is,

$$S=R\vartriangle Q,$$

Let:

$$\vartriangle=\begin{pmatrix}0\\ 1\\ -\frac{35}{48}\\ -\frac{1}{16}\end{pmatrix}.$$

Hence the diagonal matrix of eigen value can be written as,

$$\Delta^e = \begin{pmatrix} (0)^e & 0 & 0 & 0 \\ 0 & (1)^e & 0 & 0 \\ 0 & 0 & (-\frac{1}{35})^e & 0 \\ 0 & 0 & 0 & (-\frac{1}{16})^e \end{pmatrix}.$$

Therefore,

$$\lim_{e\to\infty} (\Delta)^e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let the matrix of eigen vectors corresponding to the eigen value is,

$$Q = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & \frac{1}{18} & -\frac{1}{35} \\ 1 & 1 & \frac{1}{18} & \frac{1}{35} \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

As we know that eigen decomposition of S is,

$$S = R \vartriangle Q,$$

And inverse matrix Q^{-1} is,

$$Q^{-1} = \begin{pmatrix} \frac{1}{70} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{70} \\ -\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34} \\ \frac{9}{17} & -\frac{9}{17} & \frac{9}{17} & \frac{9}{17} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Since

$$\lim_{e\to\infty} (\S)^e = R \;\; Q \lim_{e\to\infty} (\Delta)^e,$$

Implies that

$$Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}0 & 1 & 1 & -1\\ -1 & 1 & \frac{1}{18} & -\frac{1}{35}\\ 1 & 1 & \frac{1}{18} & \frac{1}{35}\\ 0 & 1 & 1 & 1\end{pmatrix}.\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Again implies that

$$Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

More implies.

$$Q^{-1}\ Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{70} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{70}\\ -\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\\ \frac{9}{17} & -\frac{9}{17} & \frac{9}{17} & \frac{9}{17}\\ -\frac{1}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}$$

$$\begin{pmatrix}h_0^\infty\\ h_1^\infty\\ h_2^\infty\\ h_3^\infty\end{pmatrix}=\begin{pmatrix}-\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\\ -\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\\ -\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\\ -\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\end{pmatrix}\begin{pmatrix}h_0^0\\ h_1^0\\ h_2^0\\ h_3^0\end{pmatrix}$$

$$\lim_{e\to\infty}(\S)^e=R\ \ Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}-\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\\ -\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\\ -\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\\ -\frac{1}{34} & \frac{9}{17} & \frac{9}{17} & -\frac{1}{34}\end{pmatrix}.$$

This implies that limit stencils.

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$$\lim_{e\to\infty}(\S)^e = \left(-\frac{1}{34} \quad \frac{9}{17} \quad \frac{9}{17} \quad -\frac{1}{34}\right).$$

Limit stencils of a 4-point ternary subdivision scheme at $M_{\frac{1}{3}}$

Theorem 27: To find the limit stencil of parametric 4-point ternary subdivision scheme.

Proof. We will put $m = 0$ in the given subdivision scheme and we get,

$$h_{-1}^{t+1} = \frac{7}{48}h_{-1}^t + \frac{13}{24}h_0^t + \frac{5}{16}h_1^t + 0,$$

$$h_0^{t+1} = \frac{5}{144}h_{-1}^t + \frac{67}{144}h_0^t + \frac{67}{144}h_1^t + \frac{5}{144}h_2^t,$$

$$h_1^{t+1} = 0 + \frac{5}{16}h_0^t + \frac{13}{24}h_1^t + \frac{7}{48}h_2^t.$$

We will put $m = 1$ in the given subdivision scheme and we get,

$$h_2^{t+1} = \frac{7}{48}h_0^t + \frac{13}{24}h_1^t + \frac{5}{16}h_2^t + 0,$$

$$h_3^{t+1} = \frac{5}{144}h_0^t + \frac{67}{144}h_1^t + \frac{67}{144}h_2^t + \frac{5}{144}h_3^t,$$

$$h_4^{t+1} = 0 + \frac{5}{16}h_1^t + \frac{13}{24}h_2^t + \frac{7}{48}h_3^t.$$

The subdivision matrix of the scheme defined by is

$$\begin{pmatrix} h_0^{t+1} \\ h_1^{t+1} \\ h_2^{t+1} \\ h_3^{t+1} \end{pmatrix} = \begin{pmatrix} \frac{5}{144} & \frac{67}{144} & \frac{67}{144} & \frac{5}{144} \\ 0 & \frac{5}{16} & \frac{13}{24} & \frac{7}{48} \\ \frac{7}{48} & \frac{13}{24} & \frac{5}{16} & 0 \\ \frac{5}{144} & \frac{67}{144} & \frac{67}{144} & \frac{7}{48} \end{pmatrix} \begin{pmatrix} h_0^t \\ h_1^t \\ h_2^t \\ h_3^t \end{pmatrix}$$

It can also be written as.

$$h^{t+1} = Sh^t.$$

Now we find subdivision matrix "S" of given scheme is

$$S = \begin{pmatrix} \frac{5}{144} & \frac{67}{144} & \frac{67}{144} & \frac{5}{144} \\ 0 & \frac{5}{16} & \frac{13}{24} & \frac{7}{48} \\ \frac{7}{48} & \frac{13}{24} & \frac{5}{16} & 0 \\ \frac{5}{144} & \frac{67}{144} & \frac{67}{144} & \frac{7}{48} \end{pmatrix}.$$

we find eigen value's and eigen vector's of matrix "S" Now we shall compute the eigen decomposition of subdivision scheme. For this

$$\lambda = \begin{pmatrix} 0 \\ 1 \\ -\frac{11}{48} \\ -\frac{11}{144} \end{pmatrix}.$$

The matrix of eigen vector conversing to eigen value is has

$$Q = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{21}{134} & \frac{7}{11} \\ 1 & 1 & -\frac{21}{134} & -\frac{7}{11} \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

As we know that eigen decomposition of S is,

$$S = R \vartriangle Q,$$

Let:

$$\vartriangle = \begin{pmatrix} 0 \\ 1 \\ -\frac{11}{48} \\ -\frac{11}{144} \end{pmatrix}.$$

Hence the diagonal matrix of eigen value can be written as,

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$$\Delta^{e}=\begin{pmatrix}(0)^{e} & 0 & 0 & 0\\ 0 & (1)^{e} & 0 & 0\\ 0 & 0 & (-\frac{11}{48})^{e} & 0\\ 0 & 0 & 0 & (-\frac{11}{144})^{e}\end{pmatrix}.$$

Therefore,

$$\lim_{e\to\infty}(\Delta)^{e}=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Let the matrix of eigen vectors corresponding to the eigen value is,

$$Q=\begin{pmatrix}0 & 1 & 1 & -1\\ -1 & 1 & -\frac{21}{134} & \frac{7}{11}\\ 1 & 1 & -\frac{21}{134} & -\frac{7}{11}\\ 0 & 1 & 1 & 1\end{pmatrix}.$$

And inverse matrix Q^{-1} is,

$$Q^{-1}=\begin{pmatrix}-\frac{7}{22} & \frac{-1}{2} & \frac{1}{2} & \frac{7}{22}\\ \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310}\\ \frac{67}{155} & -\frac{67}{155} & -\frac{67}{155} & \frac{67}{155}\\ \frac{-1}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}.$$

Since

$$\lim_{e\to\infty}(\S)^{e}=R\ \ Q\lim_{e\to\infty}(\Delta)^{e},$$

Implies that

$$Q\lim_{e\to\infty}(\Delta)^{e}=\begin{pmatrix}0 & 1 & 1 & -1\\ -1 & 1 & -\frac{21}{134} & \frac{7}{11}\\ 1 & 1 & -\frac{21}{134} & -\frac{7}{11}\\ 0 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Again implies that

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$$Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

More implies.

$$Q^{-1}\ Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{7}{22} & \frac{-1}{2} & \frac{1}{2} & \frac{7}{22} \\ \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \\ \frac{67}{155} & -\frac{67}{155} & -\frac{67}{155} & \frac{67}{155} \\ \frac{-1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} h_0^\infty \\ h_1^\infty \\ h_2^\infty \\ h_3^\infty \end{pmatrix} = \begin{pmatrix} \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \\ \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \\ \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \\ \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \end{pmatrix} \begin{pmatrix} h_0^0 \\ h_1^0 \\ h_2^0 \\ h_3^0 \end{pmatrix}$$

$$\lim_{e\to\infty}(\S)^e = R\ \ Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \\ \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \\ \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \\ \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \end{pmatrix}.$$

This implies that limit stencils.

$$\lim_{e\to\infty}(\S)^e = \begin{pmatrix} \frac{21}{310} & \frac{67}{155} & \frac{67}{155} & \frac{21}{310} \end{pmatrix}.$$

Limit stencils of a 4-point ternary subdivision scheme at $M_{\frac{1}{4}}$

Theorem 28: To find the limit stencil of parametric 4-point ternary subdivision scheme.

Proof. We will put $m = 0$ in the given subdivision scheme and we get,

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$$h_{-1}^{t+1}=\frac{23}{96}h_{-1}^{t}+\frac{35}{96}h_{0}^{t}+\frac{37}{96}h_{1}^{t}+\frac{1}{96}h_{2}^{t},$$
$$h_{0}^{t+1}=\frac{1}{12}h_{-1}^{t}+\frac{5}{12}h_{0}^{t}+\frac{5}{12}h_{1}^{t}+\frac{1}{12}h_{2}^{t},$$
$$h_{1}^{t+1}=\frac{1}{96}h_{-1}^{t}+\frac{37}{96}h_{0}^{t}+\frac{35}{96}h_{1}^{t}+\frac{23}{96}h_{2}^{t}.$$

We will put $m=1$ in the given subdivision scheme and we get,

$$h_{2}^{t+1}=\frac{23}{96}h_{0}^{t}+\frac{35}{96}h_{1}^{t}+\frac{37}{96}h_{2}^{t}+\frac{1}{96}h_{3}^{t},$$
$$h_{3}^{t+1}=\frac{1}{12}h_{0}^{t}+\frac{5}{12}h_{1}^{t}+\frac{5}{12}h_{2}^{t}+\frac{1}{12}h_{3}^{t},$$
$$h_{4}^{t+1}=\frac{1}{96}h_{0}^{t}+\frac{37}{96}h_{1}^{t}+\frac{35}{96}h_{2}^{t}+\frac{23}{96}h_{3}^{t}.$$

The subdivision matrix of the scheme defined by is

$$\begin{pmatrix} h_0^{t+1}\\ h_1^{t+1}\\ h_2^{t+1}\\ h_3^{t+1}\end{pmatrix}=\begin{pmatrix} \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12}\\ \frac{1}{96} & \frac{37}{96} & \frac{35}{96} & \frac{23}{96}\\ \frac{23}{96} & \frac{35}{96} & \frac{37}{96} & -\frac{1}{48}\\ \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12}\end{pmatrix}\begin{pmatrix} h_0^{t}\\ h_1^{t}\\ h_2^{t}\\ h_3^{t}\end{pmatrix}$$

It can also be written as.

$$h^{t+1}=Sh^{t}.$$

Now we find subdivision matrix "S" of given scheme is

$$S=\begin{pmatrix} \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12}\\ \frac{1}{96} & \frac{37}{96} & \frac{35}{96} & \frac{23}{96}\\ \frac{23}{96} & \frac{35}{96} & \frac{37}{96} & -\frac{1}{48}\\ \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12}\end{pmatrix}$$

we find eigen value’s and eigen vector’s of matrix "S" Now we shall compute the eigen decomposition of subdivision scheme. For this

$$\lambda = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{48} \\ -\frac{1}{12} \end{pmatrix}.$$

The matrix of eigen vector conversing to eigen value is has

$$Q = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{3}{10} & -11 \\ 1 & 1 & -\frac{3}{10} & 11 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

As we know that eigen decomposition of S is,

$$S = R \vartriangle Q,$$

Let:

$$\vartriangle = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{48} \\ -\frac{1}{12} \end{pmatrix}.$$

Hence the diagonal matrix of eigen value can be written as,

$$\vartriangle^e = \begin{pmatrix} (0)^e & 0 & 0 & 0 \\ 0 & (1)^e & 0 & 0 \\ 0 & 0 & (\frac{1}{48})^e & 0 \\ 0 & 0 & 0 & (-\frac{1}{12})^e \end{pmatrix}.$$

Therefore,

$$\lim_{e\to\infty} (\Delta)^e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let the matrix of eigen vectors corresponding to the eigen value is,

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$$Q = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{3}{10} & -11 \\ 1 & 1 & -\frac{3}{10} & 11 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

And inverse matrix Q^{-1} is,

$$Q^{-1} = \begin{pmatrix} \frac{11}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{11}{2} \\ \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \\ \frac{5}{13} & -\frac{5}{13} & -\frac{5}{13} & \frac{5}{13} \\ \frac{-1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Since

$$\lim_{e\to\infty} (§)^e = R \ \ Q \lim_{e\to\infty} (\Delta)^e,$$

Implies that

$$Q \lim_{e\to\infty} (\Delta)^e = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{3}{10} & -11 \\ 1 & 1 & -\frac{3}{10} & 11 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again implies that

$$Q \lim_{e\to\infty} (\Delta)^e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

More implies.

$$Q^{-1} \ \ Q \lim_{e\to\infty} (\Delta)^e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{11}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{11}{2} \\ \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \\ \frac{5}{13} & -\frac{5}{13} & -\frac{5}{13} & \frac{5}{13} \\ \frac{-1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} h_0^\infty \\ h_1^\infty \\ h_2^\infty \\ h_3^\infty \end{pmatrix} = \begin{pmatrix} \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \\ \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \\ \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \\ \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \end{pmatrix} \begin{pmatrix} h_0^0 \\ h_1^0 \\ h_2^0 \\ h_3^0 \end{pmatrix}$$

$$\lim_{e\to\infty}(\S)^e = R\ \ Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \\ \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \\ \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \\ \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \end{pmatrix}.$$

This implies that limit stencils.

$$\lim_{e\to\infty}(\S)^e = \begin{pmatrix} \frac{3}{26} & \frac{5}{13} & \frac{5}{13} & \frac{3}{26} \end{pmatrix}.$$

Limit stencils of a 4-point ternary subdivision scheme at $M_{\frac{1}{5}}$

Theorem 29: To find the limit stencil of parametric 4-point ternary subdivision scheme.

Proof. We will put $m = 0$ in the given subdivision scheme and we get,

$$h_{-1}^{t+1} = \frac{71}{240}h_{-1}^t + \frac{31}{120}h_0^t + \frac{103}{240}h_1^t + \frac{1}{60}h_2^t,$$

$$h_0^{t+1} = \frac{9}{80}h_{-1}^t + \frac{31}{60}h_0^t + \frac{31}{60}h_1^t + \frac{9}{80}h_2^t,$$

$$h_1^{t+1} = \frac{1}{60}h_{-1}^t + \frac{103}{240}h_0^t + \frac{31}{120}h_1^t + \frac{71}{240}h_2^t.$$

We will put $m = 1$ in the given subdivision scheme and we get,

$$h_2^{t+1} = \frac{71}{240}h_0^t + \frac{31}{120}h_1^t + \frac{103}{240}h_2^t + \frac{1}{60}h_3^t,$$

$$h_3^{t+1} = \frac{9}{80}h_0^t + \frac{31}{60}h_1^t + \frac{31}{60}h_2^t + \frac{9}{80}h_3^t,$$

$$h_4^{t+1} = \frac{1}{60}h_0^t + \frac{103}{240}h_1^t + \frac{31}{120}h_2^t + \frac{71}{240}h_3^t$$

The subdivision matrix of the scheme defined by is

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$$\begin{pmatrix} h_0^{t+1} \\ h_1^{t+1} \\ h_2^{t+1} \\ h_3^{t+1} \end{pmatrix} = \begin{pmatrix} \frac{9}{80} & \frac{31}{60} & \frac{31}{60} & \frac{9}{80} \\ \frac{1}{60} & \frac{103}{240} & \frac{31}{120} & \frac{71}{240} \\ \frac{71}{240} & \frac{31}{120} & \frac{103}{240} & \frac{1}{60} \\ \frac{9}{80} & \frac{31}{60} & \frac{31}{60} & \frac{9}{80} \end{pmatrix} \begin{pmatrix} h_0^{t} \\ h_1^{t} \\ h_2^{t} \\ h_3^{t} \end{pmatrix}.$$

It can also be written as.

$$h^{t+1} = Sh^t.$$

Now we find subdivision matrix "S" of given scheme is

$$S = \begin{pmatrix} \frac{9}{80} & \frac{31}{60} & \frac{31}{60} & \frac{9}{80} \\ \frac{1}{60} & \frac{103}{240} & \frac{31}{120} & \frac{71}{240} \\ \frac{71}{240} & \frac{31}{120} & \frac{103}{240} & \frac{1}{60} \\ \frac{9}{80} & \frac{31}{60} & \frac{31}{60} & \frac{9}{80} \end{pmatrix}.$$

we find eigen value’s and eigen vector’s of matrix "S" Now we shall compute the eigen decomposition of subdivision scheme. For this

$$\lambda = \begin{pmatrix} 0 \\ 1 \\ \frac{41}{240} \\ -\frac{7}{80} \end{pmatrix}.$$

The matrix of eigen vector conversing to eigen value is has

$$Q = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{25}{62} & -\frac{67}{41} \\ 1 & 1 & -\frac{25}{62} & \frac{67}{41} \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

As we know that eigen decomposition of S is,

$$S = R \vartriangle Q,$$

Let:

$$\Delta = \begin{pmatrix} 0 \\ 1 \\ \frac{41}{240} \\ -\frac{7}{80} \end{pmatrix}.$$

Hence the diagonal matrix of eigen value can be written as,

$$\Delta^e = \begin{pmatrix} (0)^e & 0 & 0 & 0 \\ 0 & (1)^e & 0 & 0 \\ 0 & 0 & (\frac{41}{240})^e & 0 \\ 0 & 0 & 0 & (-\frac{7}{80})^e \end{pmatrix}.$$

Therefore,

$$\lim_{e\to\infty} (\Delta)^e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let the matrix of eigen vectors corresponding to the eigen value is,

$$Q = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{25}{62} & -\frac{67}{41} \\ 1 & 1 & -\frac{25}{62} & \frac{67}{41} \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

And inverse matrix Q^{-1} is,

$$Q^{-1} = \begin{pmatrix} \frac{67}{82} & \frac{-1}{2} & \frac{1}{2} & -\frac{67}{82} \\ \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \\ \frac{31}{87} & -\frac{31}{87} & -\frac{31}{87} & \frac{31}{87} \\ \frac{-1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Since

$$\lim_{e\to\infty} (\S)^e = R \;\; Q \lim_{e\to\infty} (\Delta)^e,$$

Implies that

$$Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{25}{62} & -\frac{67}{41} \\ 1 & 1 & -\frac{25}{62} & \frac{67}{41} \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again implies that

$$Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

More implies.

$$Q^{-1}\ Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{67}{82} & \frac{-1}{2} & \frac{1}{2} & -\frac{67}{82} \\ \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \\ \frac{31}{87} & -\frac{31}{87} & -\frac{31}{87} & \frac{31}{87} \\ \frac{-1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} h_0^\infty \\ h_1^\infty \\ h_2^\infty \\ h_3^\infty \end{pmatrix} = \begin{pmatrix} \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \\ \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \\ \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \\ \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \end{pmatrix}\begin{pmatrix} h_0^0 \\ h_1^0 \\ h_2^0 \\ h_3^0 \end{pmatrix}$$

$$\lim_{e\to\infty}(\S)^e = R\ \ Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \\ \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \\ \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \\ \frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174} \end{pmatrix}.$$

This implies that limit stencils.

$$\lim_{e\to\infty}(\S)^e=\begin{pmatrix}\frac{25}{174} & \frac{31}{87} & \frac{31}{87} & \frac{25}{174}\end{pmatrix}.$$

Limit stencils of a 4-point ternary subdivision scheme at $M_{\frac{3}{10}}$

Theorem 30: To find the limit stencil of parametric 4-point ternary subdivision scheme.

Proof. We will put $m=0$ in the given subdivision scheme and we get,

$$h_{-1}^{t+1}=\frac{11}{60}h_{-1}^t+\frac{113}{240}h_0^t+\frac{41}{120}h_1^t+\frac{1}{240}h_2^t,$$
$$h_0^{t+1}=\frac{13}{240}h_{-1}^t+\frac{107}{240}h_0^t+\frac{107}{240}h_1^t+\frac{13}{240}h_2^t,$$
$$h_1^{t+1}=\frac{1}{240}h_{-1}^t+\frac{41}{120}h_0^t+\frac{113}{240}h_1^t+\frac{11}{60}h_2^t.$$

We will put $m=1$ in the given subdivision scheme and we get,

$$h_2^{t+1}=\frac{11}{60}h_0^t+\frac{113}{240}h_1^t+\frac{41}{120}h_2^t+\frac{1}{240}h_3^t,$$
$$h_3^{t+1}=\frac{13}{240}h_{-0}^t+\frac{107}{240}h_1^t+\frac{107}{240}h_2^t+\frac{13}{240}h_3^t,$$
$$h_4^{t+1}=\frac{1}{240}h_0^t+\frac{41}{120}h_1^t+\frac{113}{240}h_2^t+\frac{11}{60}h_3^t.$$

The subdivision matrix of the scheme defined by is

$$\begin{pmatrix}h_0^{t+1}\\h_1^{t+1}\\h_2^{t+1}\\h_3^{t+1}\end{pmatrix}=\begin{pmatrix}\frac{13}{240} & \frac{107}{240} & \frac{107}{240} & \frac{13}{240}\\ \frac{1}{240} & \frac{41}{120} & \frac{1113}{240} & \frac{11}{60}\\ \frac{11}{60} & \frac{13}{240} & \frac{41}{120} & \frac{1}{240}\\ \frac{13}{240} & \frac{107}{240} & \frac{107}{240} & \frac{13}{240}\end{pmatrix}\begin{pmatrix}h_0^t\\h_1^t\\h_2^t\\h_3^t\end{pmatrix}.$$

It can also be written as.

$$h^{t+1}=Sh^t.$$

Now we find subdivision matrix "S" of given scheme is

$$S=\begin{pmatrix}\frac{13}{240}&\frac{107}{240}&\frac{107}{240}&\frac{13}{240}\\\frac{1}{240}&\frac{41}{120}&\frac{1113}{240}&\frac{11}{60}\\\frac{11}{60}&\frac{13}{240}&\frac{41}{120}&\frac{1}{240}\\\frac{13}{240}&\frac{107}{240}&\frac{107}{240}&\frac{13}{240}\end{pmatrix}.$$

we find eigen value's and eigen vector's of matrix "S" Now we shall compute the eigen decomposition of subdivision scheme. For this

$$\lambda=\begin{pmatrix}0\\1\\-\frac{31}{240}\\-\frac{19}{240}\end{pmatrix}.$$

The matrix of eigen vector conversing to eigen value is has

$$Q=\begin{pmatrix}0&1&1&-1\\-1&1&-\frac{45}{214}&\frac{43}{31}\\1&1&-\frac{45}{214}&-\frac{43}{31}\\0&1&1&1\end{pmatrix}.$$

As we know that eigen decomposition of S is,

$$S=R\,\triangle\,Q,$$

Let:

$$\triangle=\begin{pmatrix}0\\1\\-\frac{31}{240}\\-\frac{19}{240}\end{pmatrix}.$$

Hence the diagonal matrix of eigen value can be written as,

$$\triangle^{e}=\begin{pmatrix}(0)^{e}&0&0&0\\0&(1)^{e}&0&0\\0&0&(-\frac{31}{240})^{e}&0\\0&0&0&(-\frac{19}{240})^{e}\end{pmatrix}.$$

Therefore,

$$\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let the matrix of eigen vectors corresponding to the eigen value is,

$$Q = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{45}{214} & \frac{43}{31} \\ 1 & 1 & -\frac{45}{214} & -\frac{43}{31} \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

And inverse matrix Q^{-1} is,

$$Q^{-1} = \begin{pmatrix} -\frac{43}{62} & \frac{-1}{2} & \frac{1}{2} & \frac{43}{62} \\ \frac{45}{518} & \frac{107}{259} & \frac{107}{259} & \frac{45}{518} \\ \frac{107}{259} & -\frac{107}{259} & -\frac{107}{259} & \frac{107}{259} \\ \frac{-1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Since

$$\lim_{e\to\infty}(\S)^e = R\ \ Q\lim_{e\to\infty}(\Delta)^e,$$

Implies that

$$Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & -\frac{45}{214} & \frac{43}{31} \\ 1 & 1 & -\frac{45}{214} & -\frac{43}{31} \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again implies that

$$Q\lim_{e\to\infty}(\Delta)^e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

More implies.

$$Q^{-1}\ Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}0&1&0&0\\0&1&0&0\\0&1&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}-\frac{43}{62}&\frac{-1}{2}&\frac{1}{2}&\frac{43}{62}\\\frac{45}{518}&\frac{107}{259}&\frac{107}{259}&\frac{45}{518}\\\frac{107}{259}&-\frac{107}{259}&-\frac{107}{259}&\frac{107}{259}\\\frac{-1}{2}&0&0&\frac{1}{2}\end{pmatrix}$$

$$\begin{pmatrix}h_0^\infty\\h_1^\infty\\h_2^\infty\\h_3^\infty\end{pmatrix}=\begin{pmatrix}\frac{45}{518}&\frac{107}{259}&\frac{107}{259}&\frac{45}{518}\\\frac{45}{518}&\frac{107}{259}&\frac{107}{259}&\frac{45}{518}\\\frac{45}{518}&-\frac{107}{259}&-\frac{107}{259}&\frac{45}{518}\\\frac{45}{518}&\frac{107}{259}&\frac{107}{259}&\frac{45}{518}\end{pmatrix}\begin{pmatrix}h_0^0\\h_1^0\\h_2^0\\h_3^0\end{pmatrix}$$

$$\lim_{e\to\infty}(\S)^e=R\ \ Q\lim_{e\to\infty}(\Delta)^e=\begin{pmatrix}\frac{45}{518}&\frac{107}{259}&\frac{107}{259}&\frac{45}{518}\\\frac{45}{518}&\frac{107}{259}&\frac{107}{259}&\frac{45}{518}\\\frac{45}{518}&-\frac{107}{259}&-\frac{107}{259}&\frac{45}{518}\\\frac{45}{518}&\frac{107}{259}&\frac{107}{259}&\frac{45}{518}\end{pmatrix}.$$

This implies that limit stencils.

$$\lim_{e\to\infty}(\S)^e=\begin{pmatrix}\frac{45}{518}&\frac{107}{259}&\frac{107}{259}&\frac{45}{518}\end{pmatrix}$$

Table 5: ***Limit stencil of our Scheme at different values of parameter.***

β	Limit stencil

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$\frac{1}{2}$	$-\frac{1}{34},\frac{9}{17},\frac{9}{17},-\frac{1}{34}$
$\frac{1}{3}$	$\frac{21}{310},\frac{67}{155},\frac{67}{155},\frac{21}{310}$
$\frac{1}{4}$	$\frac{3}{26},\frac{5}{13},\frac{5}{13},\frac{3}{26}$
$\frac{1}{5}$	$\frac{25}{174},\frac{31}{87},\frac{31}{87},\frac{25}{174}$
$\frac{3}{10}$	$\frac{45}{518},\frac{107}{259},\frac{107}{259},\frac{45}{518}$

Table 6: ***Final Results***

Properties	$M_{\frac{1}{2}}$	$M_{\frac{1}{3}}$	$M_{\frac{1}{4}}$	$M_{\frac{1}{5}}$	$M_{\frac{3}{10}}$
Degree of generation	1	1	1	1	1
Continuity	C^1	C^1	C^1	C^1	C^1
Parametrization	Dual	Dual	Dual	Dual	Dual

10. Applications of A Family of 4-Points Ternary Subdivision Schemes based on Laurant Polynomial.

An application of a family of 4-Points ternary subdivision schemes based on the Laurant polynomial is given in Figure 1. Figure 1(i) is a control polygon while Figure 1(vi) is a limiting curve to the control polygon after five iterations. This is evident that the limit curve is smooth enough

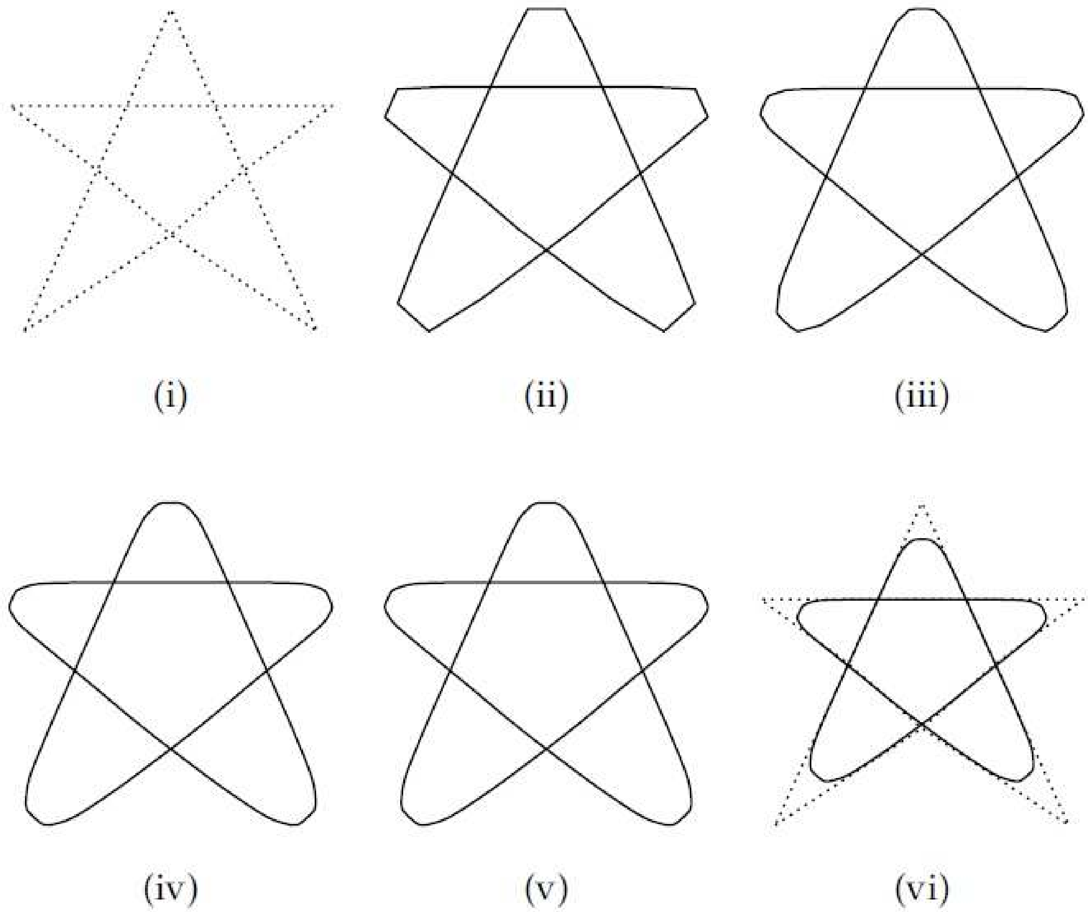

Figure 1: Applications of A Family of 4-Points Ternary Subdivision Schemes

Conclusion:

A family of 4-point ternary approximating subdivision schemes is presented. A complete analysis of the scheme is also presented. Various properties are proved for various parametric values $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, and $\frac{3}{10}$. It is concluded that a family of 4-Points ternary subdivision schemes is dual in nature and C^1 continuous. The visual representation of 4-Points ternary subdivision schemes is also presented which shows a good smoothing of curve .

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