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EFFECT OF WINKLER VARIABLE AND PASTERNAK ELASTIC FOUNDATIONS ON THE FREE VIBRATION OF FUNCTIONALLY GRADED MATERIAL PLATE.

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Abstract- Elastic foundations are widely used to study the behavior of structures under loads such as bending, buckling, and vibration. The aim of this paper is to study the free vibration of functionally graded plates (FGP) resting on a variable elastic foundation, using the refined theory of plates with four unknowns, without the need for a shear correction factor. The elastic foundation is assumed to be a two-parameter Pasternak-Winkler, with a variety of the Winkler layer along one side of the plate, but the Pasternak layer is constant. The originality of this paper lies in the analysis of the effect of a variable elastic foundation on the free vibrations of the FGP. To demonstrate the accuracy of this approach, numerical comparisons were made between the present formulation and those in the literature. The results are very good. Finally, new results are presented to show the effect of the Winkler variety. The results show that variations in the modulus of the Winkler foundation have a considerable effect on the natural frequency.

Key words: elastic foundations effect, free vibration, functionally graded plate, refined plate theory, Winkler variation.

I. INTRODUCTION:

The concept of functionally graded beams, plates, and shells resting on elastic foundations is widely used by diverse designers for bending and stability analysis. These structures are intended for many applications like aircraft, civil engineering, shipbuilding, mechanics, and various other areas. It might be of interest to inquire further into the nature of functionally graded materials, which are a relatively new composite material made by combining two distinct constituents, most often a ceramic and metal. One notable feature of these materials is that their material properties vary continuously in one or more directions of the structure. However, they have been used in many systems due to their excellent qualities, which allow for the design of lightweight structures, a high stiffness-to-weight ratio, and no interlaminar stress concentration.

There are several types of mathematical models that have been developed to describe the interactions of structure-foundation. The first simplest model for the elastic foundation in this direction has been provided by Winkler, called the one-parameter model [1], which considers the base to be made up of independent and carefully spaced linear springs. A two-parameter model by adding shear interactions between spring elements is proposed later by Pasternak [2]. Some models of elastic foundation and development of certain ideas are studied by Kerr [3]. Considerable attention has been shown in the behaviors of FGM structures on elastic foundations under various loading scenarios, which are significant for practical applications. Tounsi and his co-workers investigated the vibration of standard and sandwich FG plates resting on different elastic foundations under thermal, hydrothermal, and mechanical loads. Higher-order plate theories in conjunction with Hamilton's principle were employed to obtain the equation of motion in these works [4-9]. Considering this course of action, based on the first-order shear deformation plate theory, Yüksel and Akbaş [10] investigated the free vibration analysis of a sandwich plate made of laminated face and porous core layers resting on a Winkler-Pasternak foundation. They considered temperature and moisture dependent material properties. Moreover, the thermo-mechanical buckling behavior of FGM plates with a negative Poisson ratio resting on an elastic foundation is investigated by Mansouri and Shariyat [11]. [12-14] gave an analytic solution for the vibration analysis of the beam and plates composed of functionally graded materials resting on elastic foundation. Different plate theories in combination with Von Kármán assumptions are used herein. Zenkour [15] examined the thermoelastic bending response of a simply supported rectangular FG plate resting on two-parameter elastic foundations by using the refined sinusoidal shear deformation theory. Marouf et al. [16] also adopted a refined four-unknown theory to investigate the free vibration of power-law functionally graded plates supported by a variable elastic foundation. Yuda and Xiaoguang investigated the parametric vibration and stability of a simply supported FG plate; they used CPT and von Karman assumptions to find the stability characteristics [17]. Nonpolynomial higher-order shear and normal deformation theory are used by Gupta et al. [18] to analyze the static and dynamic features of a FG plate resting on a Pasternak foundation. The thickness stretching effect and two specific types of micromechanics models containing Voigt and Mori-Tanaka are considered in their work. Shahsavari et al. [19] determined the natural frequencies of imperfect FG porous plates resting on Winkler, Pasternak or Kerr foundations. In their articles, a new quasi-3D hyperbolic theory and different porosity distribution models are investigated.

It would seem that the majority of research on vibration has only looked at plates that are evenly supported on an elastic foundation. However, research on structures supported by variable elastic foundations is scarce in the literature. Oni and Awodola [20] evaluate the dynamic behavior of simply supported rectangular plates resting on a polynomial variable Winkler elastic foundation under moving concentrated masses. The free vibration and buckling analysis of porous FG beams resting on a variable elastic foundation were studied by Mellal et al. [21]. Pham et al. [22] employed the generalized finite element method to investigate hygro-thermal vibration of bidirectional FG porous curved beams on an elastic foundation with a variable Winkler parameter and a fixed Pasternak parameter. Ketabdari et al. [23] found the free vibration analysis of homogeneous and FG skew plates resting on variable Winkler-Pasternak elastic foundation based on the Rayleigh-Ritz method. A quasi-3D shear deformation plate theory is employed to investigate the thermodynamic response of FG sandwich plates lying on a variable elastic foundation by Bouiadjra et al. [24].

To our knowledge, there have been no studies on the free vibration of FG plates resting on variable elastic foundations. For this reason, the aim of this paper is to investigate FG plates resting on a variable elastic foundation. The elastic foundation composed of two parameters takes into account linear, parabolic, sinusoidal, cosine, and exponential variations of the Winkler foundation moduli along the direction. These plates are assumed to have variable material properties as a function of thickness. The analysis procedure is based on a refined shear deformation theory, so no shear correction factor is required. By applying Hamilton's principle, the equations of motion of FG plates resting on a variable elastic foundation are obtained and solved by Navier's method. Consequently, the eigenvalue issue can be solved to determine fundamental frequencies. The precision of the current method is validated through comparison with both the third-order shear deformation theory (TSDT) as presented in the literature and alternative shear deformation plate theories. The results reveal that differences in the Winkler foundation's modulus have a considerable effect on the natural frequency.

II. GOVERNING EQUATIONS OF FGM PLATE ON ELASTIC FOUNDATION

Take a rectangular FG plate with length a, width b, and uniform thickness h in the Cartesian coordinate system, supported by an elastic foundation; see Fig. 1.

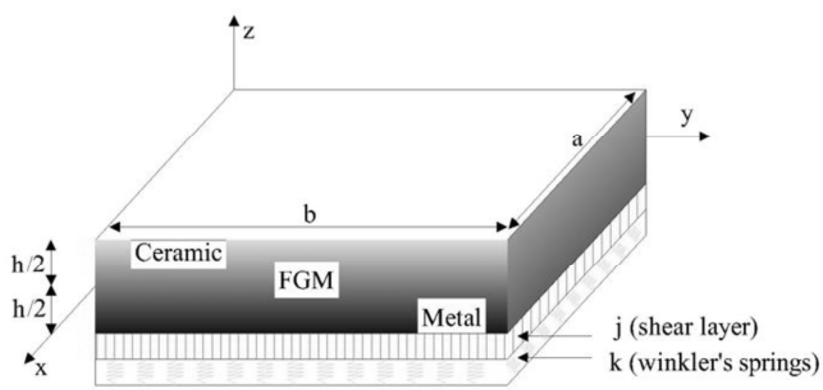

FIG. 1. Geometry of a rectangular FG plate resting on a two-parameter elastic foundation.

A. MATERIALS PROPERTIES

The properties of the constituent materials are assumed to vary continuously according to a power law in thickness and are listed in Table 1:

Table 1. Material properties used of FG plates

Material	Properties		
	E (GPa)	ν	ρ (kg/m^3)
Aluminum (Al)	70	0.3	2702
Alumina (Al_2O_3)	380	0.3	3800

(2.1) $$\Omega(z) = \Omega_m + (\Omega_c - \Omega_m)\left(\frac{1}{2} + \frac{z}{h}\right)^p$$

Where Ω represents the material property such as Young's modulus E, mass density ρ, and Poisso's ratio ν, for simplicity, the last is assumed to be constant [25-27]. $0 \leq p \leq \infty$ is the power-law index that indicates the material variation profile in the thickness.

B. CONSTITUTIVE RELATIONS

Based on a refined plate theory, the displacement fields of the present study are:

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x}$$
$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \quad (2.2)$$
$$w(x,y,z,t) = w_b(x,y,t) + w_s(x,y,t)$$

The u, v, and w are the displacements in the x, y, and z directions, respectively; u_0 and v_0 are the midplane displacements; w_b and w_s are the bending and shear components of the transverse displacement w, respectively.
The von Kármán strain of the displacement field in Eq. (2.2) are:

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} + z\begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix} + f(z)\begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix}, \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} = g(z)\begin{Bmatrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{Bmatrix}, \varepsilon_z = 0 \quad (2.3)$$

Where:

$$\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} = \begin{Bmatrix}\frac{\partial u_0}{\partial x}\\ \frac{\partial v_0}{\partial y}\\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\end{Bmatrix}, \begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix} = \begin{Bmatrix}-\frac{\partial^2 w_b}{\partial x^2}\\ -\frac{\partial^2 w_b}{\partial y^2}\\ -2\frac{\partial^2 w_b}{\partial x\partial y}\end{Bmatrix},$$
$$\begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix} = \begin{Bmatrix}-\frac{\partial^2 w_s}{\partial x^2}\\ -\frac{\partial^2 w_s}{\partial y^2}\\ -2\frac{\partial^2 w_s}{\partial x\partial y}\end{Bmatrix}, \begin{Bmatrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{Bmatrix} = \begin{Bmatrix}\frac{\partial w_s}{\partial y}\\ \frac{\partial w_s}{\partial x}\end{Bmatrix} \quad (2.4)$$
$$f(z) = z - z\,e^{-2\left(\frac{z}{h}\right)^2}, g(z) = 1 - \frac{\partial f(z)}{\partial z}$$

The linear stress–strain relationships of FG plate are:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{Bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & 0 & 0 & 0\\ Q_{21} & Q_{22} & 0 & 0 & 0\\ 0 & 0 & Q_{66} & 0 & 0\\ 0 & 0 & 0 & Q_{44} & 0\\ 0 & 0 & 0 & 0 & Q_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \quad (2.5)$$

in which Q_{ij} are the elastic constants.

$$Q_{11} = Q_{22} = \frac{E(z)}{(1-\nu^2)}\ ;\ Q_{12} = \nu Q_{11} = \frac{\nu\, E(z)}{(1-\nu^2)}\ ;\ Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\nu)} \quad (2.6)$$

The analytical form of the Hamilton's principle of FG plate can be written as

$$\int_0^t \left(\delta U_p + \delta U_f - \delta K\right)dt = 0 \quad (2.7)$$

Where δU_p, δU_f and δK are the variation of strain energy, potential energy of elastic foundation and kinetic energy, respectively, where they are:

$$\delta U_p = \int_v \sigma_{ij}\,\delta\varepsilon_{ij}dv = \int_v \left[\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz}\right]dV$$
$$\delta U_f = \int_A \{[K_w(w_b+w_s) - K_s\nabla^2(w_b+w_s)](\delta w_b + \delta w_s)\}\,dA \quad (2.8)$$
$$\delta K = \int_v \rho\,(z)[\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}]\,dV$$

Substituting Eqs. (2.6) and (2.3) into Eq. (2.5) and integrating the displacement gradients across the thickness by parts, then individually equating the coefficients of δu_0, δv_0, δw_b, and δw_s to zero, the differential equations may be derived as:

(2.9)
$$\delta u_0 : A_{11}\frac{\partial^2 u_0}{\partial x^2}+A_{66}\frac{\partial^2 u_0}{\partial y^2}+(A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\partial y}-B_{11}\frac{\partial^3 w_b}{\partial x^3}-(B_{12}+2B_{66})\frac{\partial^3 w_b}{\partial x\partial y^2}-B_{11}^s\frac{\partial^3 w_s}{\partial x^3}-(B_{12}^s+2B_{66}^s)\frac{\partial^3 w_s}{\partial x\partial y^2}=I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^3 w_b}{\partial t^2\partial x}-J_1\frac{\partial^3 w_s}{\partial t^2\partial x}$$

$$\delta v_0 : (A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x\partial y}+A_{66}\frac{\partial^2 v_0}{\partial x^2}+A_{22}\frac{\partial^2 v_0}{\partial y^2}-(B_{12}+2B_{66})\frac{\partial^3 w_b}{\partial x^2\partial y}-B_{22}\frac{\partial^3 w_b}{\partial y^3}-B_{22}^s\frac{\partial^3 w_s}{\partial y^3}-(B_{12}^s+2B_{66}^s)\frac{\partial^3 w_s}{\partial x^2\partial y}=I_0\frac{\partial^2 v_0}{\partial t^2}-I_1\frac{\partial^3 w_b}{\partial t^2\partial y}-J_1\frac{\partial^3 w_s}{\partial t^2\partial y}$$

$$\delta w_b : B_{11}\frac{\partial^3 u_0}{\partial x^3}+(B_{12}+2B_{66})\frac{\partial^3 u_0}{\partial x\partial y^2}+(B_{12}+2B_{66})\frac{\partial^3 v_0}{\partial x^2\partial y}+B_{22}\frac{\partial^3 v_0}{\partial y^3}-D_{11}\frac{\partial^4 w_b}{\partial x^4}-2(D_{12}+2D_{66})\frac{\partial^4 w_b}{\partial x^2\partial y^2}-D_{22}\frac{\partial^4 w_b}{\partial y^4}-D_{11}^s\frac{\partial^4 w_s}{\partial x^4}-2(D_{12}^s+2D_{66}^s)\frac{\partial^4 w_s}{\partial x^2\partial y^2}-D_{22}^s\frac{\partial^4 w_s}{\partial y^4}=K_w(w_b+w_s)-K_s\nabla^2(w_b+w_s)+I_0\left(\frac{\partial^2 w_b}{\partial t^2}+\frac{\partial^2 w_s}{\partial t^2}\right)+I_1\left(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y}\right)-I_2\nabla^2\frac{\partial^2 w_b}{\partial t^2}-J_2\nabla^2\frac{\partial^2 w_s}{\partial t^2}$$

$$\delta w_s : B_{11}^S\frac{\partial^3 u_0}{\partial x^3}+(B_{12}^S+2B_{66}^S)\frac{\partial^3 u_0}{\partial x\partial y^2}+(B_{12}^S+2B_{66}^S)\frac{\partial^3 v_0}{\partial x^2\partial y}+B_{22}^S\frac{\partial^3 v_0}{\partial y^3}-D_{11}^S\frac{\partial^4 w_b}{\partial x^4}-2(D_{12}^S+2D_{66}^S)\frac{\partial^4 w_b}{\partial x^2\partial y^2}-D_{22}^S\frac{\partial^4 w_b}{\partial y^4}-H_{11}^s\frac{\partial^4 w_s}{\partial x^4}-2(H_{12}^s+2H_{66}^s)\frac{\partial^4 w_s}{\partial x^2\partial y^2}-H_{22}^s\frac{\partial^4 w_s}{\partial y^4}+A_{55}^S\frac{\partial^2 w_s}{\partial x^2}+A_{44}^S\frac{\partial^2 w_s}{\partial y^2}=K_w(w_b+w_s)-K_s\nabla^2(w_b+w_s)+I_0\left(\frac{\partial^2 w_b}{\partial t^2}+\frac{\partial^2 w_s}{\partial t^2}\right)+J_1\left(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y}\right)-J_2\nabla^2\frac{\partial^2 w_b}{\partial t^2}-K_2\nabla^2\frac{\partial^2 w_s}{\partial t^2}$$

Where:

$$\begin{Bmatrix} A_{11} & B_{11} & D_{11} & B_{11}^s & D_{11}^s & H_{11}^s \\ A_{12} & B_{12} & D_{12} & B_{12}^s & D_{12}^s & H_{12}^s \\ A_{66} & B_{66} & D_{66} & B_{66}^s & D_{66}^s & H_{66}^s \end{Bmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}(1, z, z^2, f(z), zf(z), f(z)^2)\begin{Bmatrix} 1 \\ \nu \\ \frac{1-\nu}{2} \end{Bmatrix}dz$$

$$(A_{22}, B_{22}, D_{22}, B_{22}^s, D_{22}^s, H_{22}^s)=(A_{11}, B_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s)$$

(2.10)
$$A_{44}^s=A_{55}^s=\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{44}g(z)^2\,dz$$

$$(I_0, I_1, J_1, I_2, J_2, K_2)=\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}\rho(z)(1, z, f(z), z^2, zf(z), f^2(z))dz$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

The elastic foundation is hypothesized to comprise two parameters: Winkler's springs, which vary along one side of the FGP, and the Pasternak shear parameter.

(2.11)
$$f_e(x)=K_w(x)w-K_s\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$

The Winkler parameter varies according to the literature [28-32] as follows:

(2.12)
$$K_w(x)=\frac{J_1 D_i}{a^4}\begin{cases}\left[1+\xi\,\frac{x}{a}\right] & linear. \\ \left[1+\xi\left(\frac{x}{a}\right)^2\right] & parabolic \\ \left[1+\xi\,sin\left(\pi\frac{x}{a}\right)\right] & sinusoidal \\ \left[1+\xi\,cos\left(\frac{\pi}{2}\frac{x}{a}\right)\right] & cosine \\ \left[1+\xi\left(exp\left(\frac{x}{a}\right)-exp\left(\left(\frac{x}{a}\right)^\eta\right)\right)\right] & exponentially \end{cases}$$

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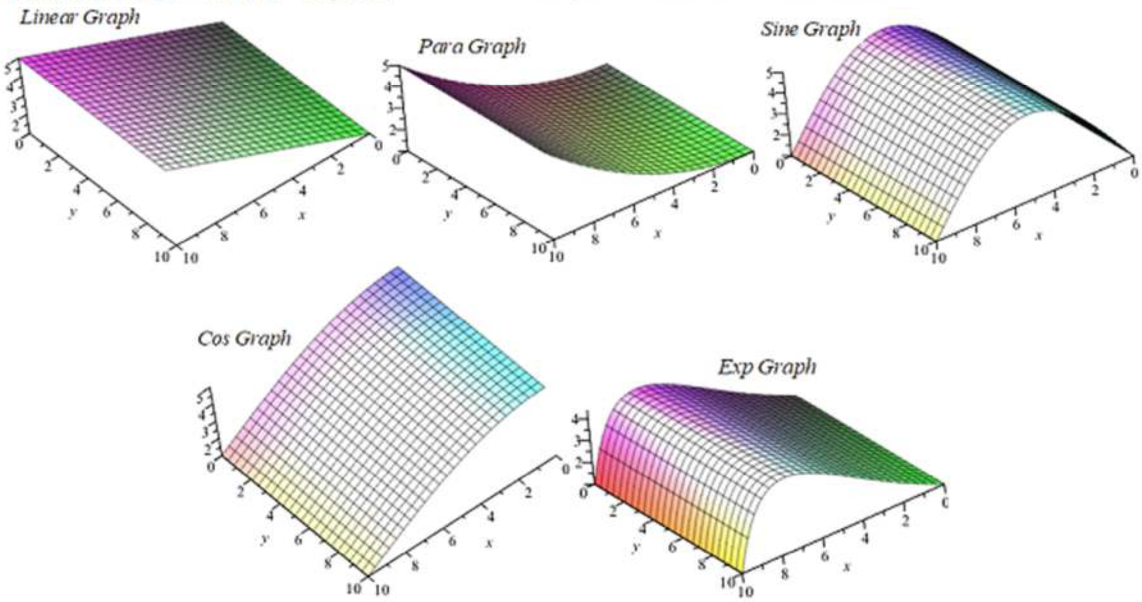

Fig.2 Several distributions of the Winkler elastic foundation along the axis x.

III. ANALYTICAL NAVIER'S SOLUTION OF FG PLATE

The determination of natural frequencies is of great importance in the design of some structural components. For the simply supported rectangular plate, the solution to the system of partial differential Eq. (2.9) is derived based on double Fourier series, with the Navier process being applied.

$$u_0(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} e^{i\omega t}\cos(\alpha x)\sin(\beta y)$$
$$v_0(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} e^{i\omega t}\sin(\alpha x)\cos(\beta y)$$
$$w_b(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn} e^{i\omega t}\sin(\alpha x)\sin(\beta y) \quad (3.1)$$
$$w_s(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} e^{i\omega t}\sin(\alpha x)\sin(\beta y)$$

$\alpha = \frac{m\pi}{a}$, $\beta = \frac{n\pi}{b}$, $i = \sqrt{-1}$, $(U_{mn}, V_{mn}, W_{bmn}, W_{smn})$ are arbitrary parameters, ω denotes the natural frequency associated with (n, m) mode.

After introducing Eqs. (3.1) into Eqs. (2.9) and using eigenvalue techniques, we obtain the following algebraic equation for the free vibration problem:

$$\left(\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{12} & s_{22} & s_{23} & s_{24} \\ s_{13} & s_{23} & s_{33} & s_{34} \\ s_{14} & s_{24} & s_{34} & s_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}\right) \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{Smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.2)$$

In which

$$s_{11} = \alpha^2 A_{11} + \beta^2 A_{66}, \quad s_{12} = \alpha\beta(A_{12} + A_{66}), \quad s_{13} = -\alpha[B_{11}\alpha^2 + (B_{12} + 2B_{66})\beta^2]$$
$$s_{22} = \alpha^2 A_{66} + \beta^2 A_{22}, \; s_{14} = -\alpha[B_{11}^s\alpha^2 + (B_{12}^s + 2B_{66}^s)\beta^2], \; s_{23} = -\beta[(B_{12} + 2B_{66})\alpha^2 + B_{22}\beta^2]$$
$$s_{24} = -\beta[(B_{12}^s + 2B_{66}^s)\alpha^2 + B_{22}^s\beta^2], \; s_{24} = -\beta[(B_{12}^s + 2B_{66}^s)\alpha^2 + B_{22}^s\beta^2]$$
$$s_{33} = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 + K_w + K_s(\alpha^2 + \beta^2), \quad (3.3)$$
$$s_{34} = (D_{11}^s\alpha^4 + 2(D_{12}^s + 2D_{66}^s)\alpha^2\beta^2 + D_{22}^s\beta^4 + K_w + K_s(\alpha^2 + \beta^2),$$
$$s_{44} = (H_{11}^s\alpha^4 + 2(H_{12}^s + 2H_{66}^s)\alpha^2\beta^2 + H_{22}^s\beta^4 + A_{55}^s\alpha^2 + A_{44}^s\beta^2 + K_w + K_s(\alpha^2 + \beta^2),$$
$$m_{11} = m_{22} = I_0, \; m_{13} = -\alpha I_1, \; m_{14} = -\alpha J_1, \; m_{23} = -\beta I_1, \; m_{24} = -\beta J_1,$$
$$m_{33} = I_0 + I_2(\alpha^2 + \beta^2), \; m_{34} = I_0 + J_2(\alpha^2 + \beta^2), \; m_{44} = I_0 + K_2(\alpha^2 + \beta^2)$$

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IV. RESULTS AND DISCUSSION

This section is concerned with a comparison of the outcomes of 2D, quasi-3D, and TSDT with those of the refined shear deformation plate theory. The present study has been undertaken for the purpose of illustrating the accuracy of current methods for predicting the free vibration of simply supported FG plates on two-parameter elastic bases through a number of numerical examples. For numerical results, an Al/Al_2O_3 plate made of aluminum (metal) and alumina (ceramic) is used. The dimensionless free vibration frequency for the first mode of FG square plates with different foundation characteristics $(\bar{k}_w, \bar{k}_s)$, thickness ratio (a/h), and power law index (p) is shown in Table 2. The findings demonstrate excellent agreement for all values of foundation parameters, thickness ratio, and power law index, thereby confirming the accuracy of the existing theory in predicting the free vibration responses of FG square plates on elastic foundations. It is noteworthy that the proposed theory yields congruent outcomes with the two-dimensional shear deformation theory presented by Zaoui et al. [33].
The non-dimensional parameters used are:

(4.1)
$$\bar{\omega} = \omega h\sqrt{\rho_m/E_m} \quad ; \quad \beta = \omega h\sqrt{\rho_c/E_c}$$
$$\bar{k}_w = k_w a^4/D_m, \quad \bar{k}_s = k_s a^2/D_m \quad \text{where} \quad D_m = E_m h^3/12(1-\nu^2)$$

Table 2. Non-dimensional fundamental frequencies $\bar{\omega} = \omega h\sqrt{\rho_m/E_m}$ of FG square plates on elastic foundation.

$(\bar{k}_w, \bar{k}_s)$	h/a	Theory	P				
			0	0.5	1	2	5
(0.0)	0.05	TSDT [34]	0.0291	0.0249	0.0227	0.0209	0.0197
		2D [33]	0.0291	0.0246	0.0222	0.0202	0.0191
		quasi-3D [33]	0.0291	0.0248	0.0226	0.0207	0.0195
		Present	0.02907	0.02464	0.02221	0.02018	0.01909
	0.2	TSDT [34]	0.4154	0.3606	0.3299	0.3016	0.2765
		2D [33]	0.4151	0.3551	0.3205	0.2892	0.2665
		quasi-3D [33]	0.4178	0.3593	0.3267	0.2968	0.2725
		Present	0.41529	0.35525	0.32064	0.28930	0.26642
(0. 100)	0.05	TSDT [34]	0.0406	0.0389	0.0382	0.0380	0.0381
		2D [33]	0.0406	0.0386	0.0378	0.0374	0.0377
		quasi-3D [33]	0.0406	0.0387	0.0380	0.0376	0.0378
		Present	0.04056	0.03863	0.03779	0.03738	0.03765
	0.2	TSDT [34]	0.6080	0.5932	0.5876	0.5861	0.5879
		2D [33]	0.6075	0.5857	0.5753	0.5694	0.5722
		quasi-3D [33]	0.6015	0.5795	0.5701	0.5652	0.5662
		Present	0.60761	0.58580	0.57534	0.56947	0.57214
(100. 0)	0.05	TSDT [34]	0.0298	0.0258	0.0238	0.0221	0.0210
		2D [33]	0.0298	0.0255	0.0233	0.0214	0.0204
		quasi-3D [33]	0.0298	0.0257	0.0236	0.0219	0.0208
		Present	0.02976	0.02554	0.02325	0.02139	0.02044
	0.2	TSDT [34]	0.4273	0.3758	0.3476	0.3219	0.2999
		2D [33]	0.4269	0.3702	0.3381	0.3097	0.2900
		quasi-3D [33]	0.4290	0.3737	0.3433	0.3161	0.2948
		Present	0.42713	0.37041	0.33822	0.30971	0.28988
(100,100)	0.05	TSDT [34]	0.0411	0.0395	0.0388	0.0386	0.0388
		2D [33]	0.0411	0.0392	0.0384	0.0381	0.0384

		quasi-3D [33]	0.0411	0.0393	0.0386	0.0383	0.0385
		Present	0.04106	0.03921	0.03841	0.03805	0.03835
	0.2	TSDT [34]	0.6162	0.6026	0.5978	0.5970	0.5993
		2D [33]	0.6156	0.5950	0.5852	0.5800	0.5833
		quasi-3D [33]	0.6093	0.5884	0.5797	0.5754	0.5770
		Present	0.61575	0.59509	0.58528	0.58003	0.58332

Tables 3 and 4 display the non-dimensional vibratory frequency of simply supported FG plates on different types of elastic foundations for various thickness-to-side ratios ($h/a = 0.05$ and 0.2), five variable foundation parameters $(\bar{k}(x)_w, \bar{k}_s)$, power law index (p), side to length (a/b), and variation parameter ($\xi = 10$ and 20). It can be seen that the non-dimensional frequency of FG plates decreases with increasing power law index (p). Increasing the ratio (h/a), variation parameter (ξ), and variable foundation parameters $(\bar{k}(x)_w, \bar{k}_s)$ increases the non-dimensional frequency; the effects of the Winkler parameters are greater than those of the Pasternak parameters.

Table 3. Non-dimensional fundamental frequencies $\bar{\omega} = \omega h\sqrt{\rho_m/E_m}$ of FG square plates on variable elastic foundation.

$(\bar{k}_w, \bar{k}_s)$	h/a	Variation forms	$\xi = 10$, P = 0	1	5	10	$\xi = 20$, P = 0	1	5	10
(100. 0)	0.05	lin	0.03299	0.02788	0.02616	0.02590	0.03593	0.03185	0.03084	0.03075
		para	0.03142	0.02567	0.02348	0.02310	0.03299	0.02788	0.02616	0.02590
		sin	0.03593	0.03185	0.03084	0.03075	0.04119	0.03857	0.03853	0.03866
		cos	0.03424	0.02959	0.02819	0.02801	0.03819	0.03479	0.03423	0.03425
		exp	0.03126	0.02544	0.02319	0.02279	0.03268	0.02745	0.02564	0.02536
		fixed	0.02976	0.02325	0.02044	0.01991	0.02976	0.02325	0.02044	0.01991
	0.2	lin	0.48196	0.41505	0.38625	0.38240	0.53116	0.47966	0.46279	0.46208
		para	0.45537	0.37858	0.34152	0.33546	0.48196	0.41505	0.38625	0.38240
		sin	0.53116	0.47966	0.46279	0.46208	0.61786	0.58786	0.58618	0.58950
		cos	0.50293	0.44294	0.41967	0.41728	0.56869	0.52719	0.51759	0.51877
		exp	0.45257	0.37469	0.33664	0.33030	0.47668	0.40790	0.37758	0.37336
		fixed	0.42713	0.33822	0.28988	0.28069	0.42713	0.33822	0.28988	0.28069
(100. 100)	0.05	lin	0.04346	0.04138	0.04168	0.04189	0.04573	0.04415	0.04476	0.04505
		para	0.04227	0.03992	0.04005	0.04022	0.04346	0.04138	0.04168	0.04189
		sin	0.04573	0.04415	0.04476	0.04505	0.04996	0.04922	0.05037	0.05079
		cos	0.04441	0.04255	0.04298	0.04323	0.04753	0.04632	0.04717	0.04751
		exp	0.04215	0.03977	0.03988	0.04005	0.04322	0.04109	0.04135	0.04156
		Fixed	0.04106	0.03841	0.03835	0.03848	0.04106	0.03841	0.03835	0.03848
	0.2	Lin	0.65495	0.63265	0.63613	0.64085	0.69192	0.67664	0.68465	0.64091
		Para	0.63563	0.60941	0.61031	0.61432	0.65495	0.63265	0.63613	0.64085
		Sin	0.69192	0.67664	0.68465	0.64091	0.76045	0.75691	0.70310	0.64091
		Cos	0.67049	0.65123	0.65669	0.64091	0.72112	0.71099	0.70310	0.64091
		Exp	0.63364	0.60702	0.60761	0.61156	0.65110	0.62799	0.63097	0.63557
		Fixed	0.61575	0.58528	0.58332	0.58655	0.61575	0.58528	0.58332	0.58655

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Figures 3, 4, 5, and 6 show how several forms of variable elastic foundations (parabolic, linear, sinusoidal, cosine, and exponential) affect non-dimensional fundamental frequencies for a simply supported FG plate. Viewing Figs. 4, 5, and 6, it is evident that for any central deflection, the fundamental frequency increases in accordance with the order of elastic foundation forms: sinusoidal, cosine, linear, parabolic, and exponential. In figure 3 it can be seen that the dynamic deflection of the center of the FG plate is clearly affected by the variation of elastic foundations. The shape of the curves is similar to that of the foundation's distribution function.

Figure 4 shows the non-dimensional fundamental frequencies (ϖ) of square FGM plates as a function of ξ. The frequencies of elastic foundations increase as the parameter ξ increases. However, fundamental frequencies remain constant for sine and cosine foundations with ξ values above 25 and 35, respectively. For each change in the elastic foundation, Figure 5 shows that the fundamental frequencies decline intensely when the plate is thick ($a/h \leq 20$) and slowly for the thin plates. Depending on whether the elastic foundation is fixed, exponential, parabolic, linear, cosine, or sinusoidal, fundamental frequencies rise accordingly. The results for different length-to-side ratios (a/b) are plotted in Figure 6, where the non-dimensional frequency increases as the ratio (a/b) increases.

Table 4. Non-dimensional fundamental frequencies $\beta = \omega h\sqrt{\rho_c/E_c}$ of FG plates on variable elastic foundation ($h/a = 0.15$).

$(\bar{k}_w, \bar{k}_s)$	a/b	Variation forms	$\xi = 10$				$\xi = 20$			
			P							
			0	1	5	10	0	1	5	10
(100. 0)	0.5	lin	0.18353	0.15034	0.13280	0.12335	0.23749	0.19565	0.17334	0.16054
		para	0.14940	0.12148	0.10688	0.09965	0.18353	0.15034	0.13280	0.12335
		sin	0.23749	0.19565	0.17334	0.16054	0.31911	0.26381	0.21219	0.19342
		cos	0.20759	0.17058	0.15093	0.13996	0.27426	0.22640	0.20076	0.18576
		exp	0.14549	0.11815	0.10388	0.09691	0.17713	0.14494	0.12796	0.11892
		fixed	0.10468	0.08313	0.07214	0.06813	0.10468	0.08313	0.07214	0.06813
	2	lin	0.32840	0.25879	0.21791	0.20635	0.36028	0.28637	0.24381	0.22980
		para	0.31124	0.24382	0.20371	0.19355	0.32840	0.25879	0.21791	0.20635
		sin	0.36028	0.28637	0.24381	0.22980	0.41673	0.33473	0.28862	0.27060
		cos	0.34197	0.27056	0.22900	0.21637	0.38466	0.30734	0.26332	0.24753
		exp	0.30944	0.24225	0.20221	0.19220	0.32499	0.25582	0.21510	0.20381
		fixed	0.29307	0.22786	0.18843	0.17982	0.29307	0.22786	0.18843	0.17982
(100. 10)	0.5	lin	0.19821	0.16270	0.14388	0.13349	0.24901	0.20530	0.18194	0.16845
		para	0.16712	0.13648	0.12037	0.11197	0.19821	0.16270	0.14388	0.13349
		sin	0.24901	0.20530	0.18194	0.16845	0.32676	0.27103	0.21219	0.19342
		cos	0.22068	0.18156	0.16075	0.14898	0.28429	0.23478	0.20822	0.19262
		exp	0.16363	0.13353	0.11772	0.10955	0.19230	0.15773	0.13942	0.12941
		Fixed	0.12871	0.10384	0.09096	0.08516	0.12871	0.10384	0.09096	0.08516
	2	Lin	0.35987	0.28603	0.24349	0.22951	0.38915	0.31119	0.26689	0.25078
		Para	0.34428	0.27257	0.23089	0.21809	0.35987	0.28603	0.24349	0.22951
		Sin	0.38915	0.31119	0.26689	0.25078	0.44190	0.35617	0.30829	0.28858
		Cos	0.37227	0.29672	0.25345	0.23855	0.41183	0.33056	0.28479	0.26710
		Exp	0.34267	0.27117	0.22957	0.21689	0.35677	0.28335	0.24099	0.22724
		Fixed	0.32796	0.25841	0.21755	0.20602	0.32796	0.25841	0.21755	0.20602

*$\overline{k}_w = k_w a^4/\overline{D}$, $\overline{k_s} = k_s a^2/\overline{D}$

where $\overline{D} = h^3/12(1-\nu^2)\,[p(8+3p+p^2)E_m + 3(2+p+p^2)E_c]/[(1+p)(2+p)(3+p)]$

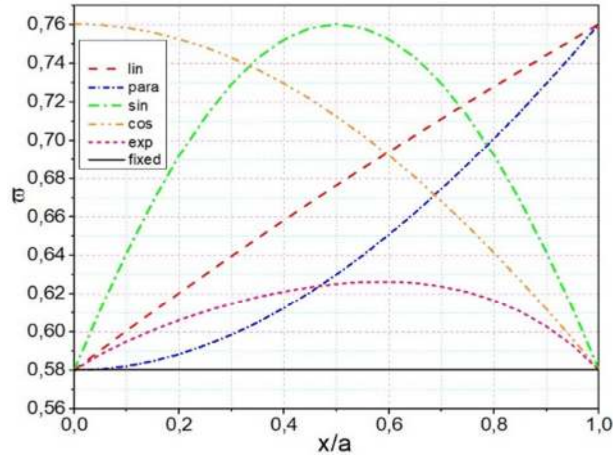

Fig. 3. variation of the dimensionless fundamental frequencies ϖ of square FG plate for various types of Winkler elastic foundation $(p = 2, a/h = 5\ \bar{k}_w = 100\ , \bar{k}_s = 100, \xi = 20)$.

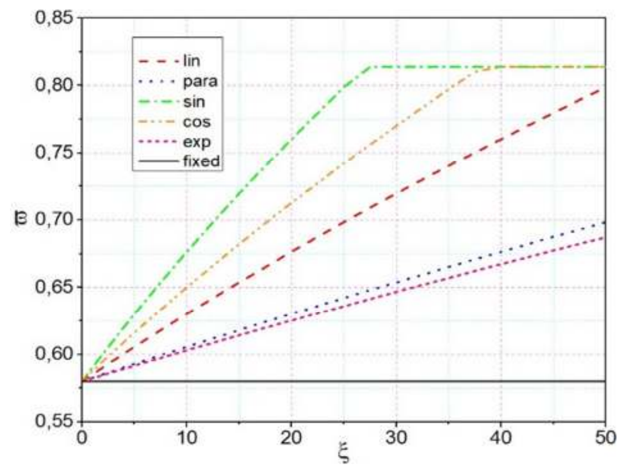

Fig. 4. Influence of parameter ξ of various types Winkler elastic foundation on the non-dimensional fundamental frequencies of square plate FG $(p = 2, a/h = 5, \bar{k}_w = \bar{k}_s = 100)$.

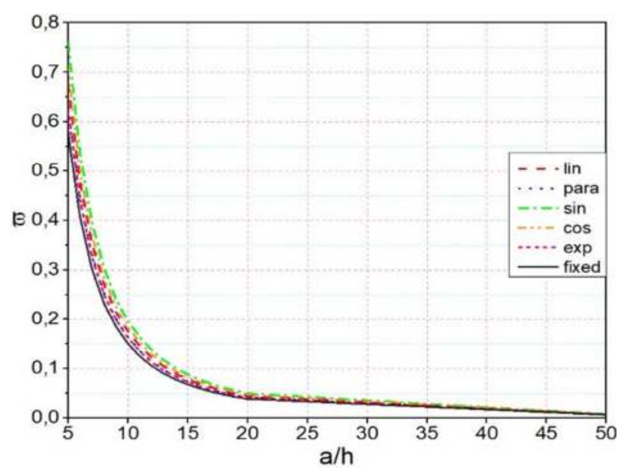

Fig. 5. Influence of various types of Winkler elastic foundation on the non-dimensional fundamental frequencies of FG square plate vs (a/h) with $(p = 2, \bar{k}_w = \bar{k}_s = 100, \xi = 20)$.

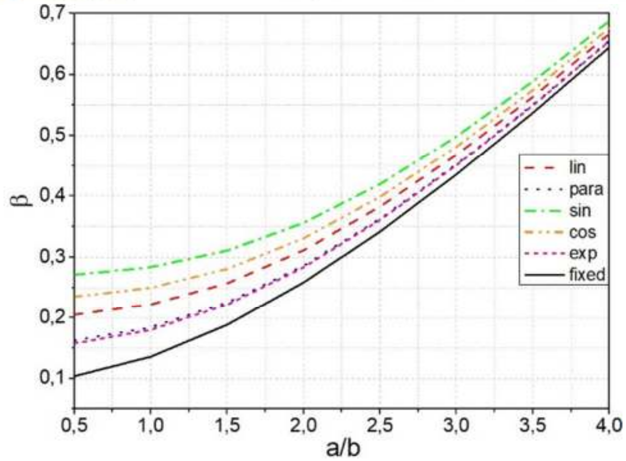

Fig. 6. Influence of various types of Winkler elastic foundation on the non-dimensional fundamental frequencies of rectangular FG plate vs (a/b) with $(p = 1, a = h/0.15, \bar{k}_w = 100, \bar{k}_s = 10, \xi = 20)$

V. CONCLUSIONS

The free vibration analysis of the functionally graded plate resting on a variable elastic foundation, assumed to vary along the plate axis according to the parabolic, linear, sinusoidal, cosine, and exponential forms, has been investigated. A mathematical model for an FGM plate is then developed in conjunction with the von Kármán, the refined plate theory, and the foundation effect using Hamilton's variational principle. The analytical method is applied to study the free vibrations of the system. The knowledge of free vibrations here is useful for studying/predicting the dynamics of plate systems. The vibration behavior of the plates is significantly influenced by material and geometric characteristics as well as variations in the elastic foundation. The conception and optimization of FGM plates requires that these system characteristics be well coordinated. The main conclusions are as follows:

1. The results presented in this work demonstrate good agreement compared to the published results.
2. The elastic foundation parameter affects the non-dimensional fundamental frequency.
3. The impact of the Winkler parameters is greater than that of the Pasternak parameters.
4. The non-dimensional frequency increases as the length-to-side ratio (a/b), the thickness-to-length ratio (h/a), the variation parameter (ξ), and the foundation parameters $(\bar{k}(x)_w, \bar{k}_s)$ increase.
5. The fundamental frequency increases in accordance with the order of the elastic foundation shapes: sinusoidal, cosine, linear, parabolic, and exponential.

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